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Formulas—Cube Root Logarithms

By

I.C.S. STAFF

FORMULAS
CUBE ROOT
LOGARITHMS

235B

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INTERNATIONAL TEXTBOOK PRESS
Scranton, Pa.

92479

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FORMULAS

ELEMENTARY OPERATIONS

USE OF SYMBOLS

RULES AND FORMULAS COMPARED

1. In the preceding Sections, and in particular in that one dealing with mensuration, frequent use is made of rules indicating how to use certain numbers for the purpose of finding an unknown length, area, or volume.

When the arithmetical processes are of a simple kind, the rule will be short and easily understood. If, however, the rule deals with several numbers that have to be combined and used in various ways, the rule becomes more complicated. It is then hard to see what must be done to apply the rule.

In order to simplify solutions a short method of expressing rules has been devised. In all arithmetical work a kind of shorthand is used. The sign + is written for the word plus; ÷ is used instead of divided by; cwt. is a symbol for 100 pounds or hundred-weight. Any one who reads a scientific or business magazine finds other shorthand expressions in which letters and the arithmetical symbols are used together. Such an expression is a short way of writing a rule, and it is called a *formula*.

A rule commonly used is: *The volume of a rectangular solid is equal to the product of its length, breadth and thickness.* This rule is often shortened by writing

$$\textit{volume} = \textit{length} \times \textit{breadth} \times \textit{thickness}$$

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Suppose the first letter of each word is used as its abbreviation. Then,

$$v = l \times b \times t$$

If it is understood that when there is no sign between two letters they are to be multiplied together, it is possible to write the expression as

$$v = lbt$$

This is a shorthand way of expressing the rule for finding the volume of a rectangular solid, and it is a *formula*.

For a longer rule, there is more advantage in writing it the short way, but in these articles simple rules must be used as illustrations before the more complicated ones are considered.

EXAMPLE.—Express the following rule as a formula, using the first letter of a word as its abbreviation: The area of a triangle equals one-half the product of its base by its height.

SOLUTION.—First shorten the rule by using arithmetical symbols to indicate the operations:

$$\text{area} = \tfrac{1}{2} \times \text{base} \times \text{height}$$

Next abbreviate each word:

$$a = \tfrac{1}{2} \times b \times h$$

The multiplication signs may be omitted and the formula becomes

$$a = \tfrac{1}{2}\, b\, h$$

EXAMPLES FOR PRACTICE

Express each of the following rules as a formula. Use the first letter of a word as its abbreviation.

1. The area of a rectangle equals the product of the base by the height.
Ans. $a = b \times h$, or $a = b\, h$

2. The side of a square equals the square root of its area.
Ans. $s = \sqrt{a}$

3. The area of a circle equals the square of its radius multiplied by 3.1416.
Ans. $a = r^2 \times 3.1416$, or $a = 3.1416\, r^2$

4. To find the interest on a given principal multiply the principal by the rate for one year and then multiply the result by the time expressed in years.
Ans. $i = p \times r \times t$, or $i = p\, r\, t$

---

2. Before formulas can be discussed, the general principles that govern letters used in them must be understood, just as the multiplication tables must be learned before it is possible to solve practical problems in which multiplication is used. In explaining the underlying principles, step by step, few practical applications can be shown, for a fair knowledge of all of the principles is necessary before their connection with formulas, which are used in the different trades and professions, can be understood.

ARITHMETICAL SIGNS AND THEIR APPLICATION

3. Signs of Aggregation.—The signs used in formulas are the ordinary signs indicative of arithmetical operations and the signs of aggregation. All these signs were explained in preceding Sections, but the signs of *aggregation* will need further explanation, as their application is extended in some directions.

The signs of **aggregation** are four in number, viz., —, ( ), [ ], and { }, respectively called the **vinculum,** the **parenthesis,** the **brackets,** and the **brace.** They are used to a great extent when it is necessary to indicate that all the numbers included by them are to be subjected to the same operation. For example, if the sum of 5 and 8 is to be multiplied by 7, any one of the four signs of aggregation may be used, but it is customary to use the *parenthesis;* thus, $(5+8)\times 7$. As already explained, the vinculum is used extensively in connection with the radical sign to indicate a root.

If *two* signs of aggregation are needed, the brackets and the parenthesis are used, so as to avoid having a parenthesis within a parenthesis, the brackets being placed outside. For example, $[(20-5)\div 3]\times 9$ means that the difference between 20 and 5 is to be divided by 3, and the result multiplied by 9.

If *three* signs of aggregation are required, the brace, brackets, and parenthesis are used, the brace being placed outside, the brackets next, and the parenthesis inside. For example, $\{[(20-5)\div 3]\times 9-21\}\div 8$ means that the quotient obtained by dividing the difference between 20 and 5 by 3 is to be multiplied by 9, and that after 21 has been subtracted from the product thus obtained, the result is to be divided by 8.

Many times a multiplication sign either preceding or following a symbol of aggregation is omitted; thus $(5+8)7$ or $7(5+8)$ means the same as $(5+8)\times 7$ or $7\times(5+8)$. In such cases the multiplication sign is understood but not written. Similarly $(2+3)\ (6+5)$ may be written instead of $(2+3)\times(6+5)$.

4. Order of Operations.—When several quantities are connected by the various signs indicating addition, subtraction, multiplication, and division, the operation indicated by the sign of *multiplication* must always be performed *first; next* in order comes the operation of *division*. Thus, $2+3\times 4$ equals 14, 3 being multiplied by 4 before adding it to 2. Similarly, $10\div 2\times 5$ equals 1, since 2×5 equals 10, and $10\div 10$ equals 1.

If this rule were not followed in the preceding examples, the results would be quite different. For instance, if in the example $2+3\times 4$ addition is performed before multiplication, as $2+3=5$, and $5\times 4=20$, the result differs from that previously found, which was 14.

In the example $10\div 2\times 5$, the quotient found by dividing 10 by 2 is 5, and 5 multiplied by 5 is 25. Performed in the correct manner the result was found to be 1.

5. In cases where several numbers are connected by using the sign of division and the plus or minus sign, the operation of division must be performed first. For example, $5-9\div 3$ is equal to 2; 9 divided by 3 gives 3 as the quotient, and $5-3=2$.

The signs of addition and subtraction are of equal value; that is, if several quantities are connected by plus and minus signs only, the indicated operations may be performed in the order in which the quantities are placed. Thus, in the example $5+7-4$, the solution may be found as $5+7=12$, and $12-4=8$. Or, $7-4=3$, and $3+5=8$.

EQUATIONS FORMED WITH NUMBERS

FUNDAMENTAL PRINCIPLES

6. Elementary Operations.—The principle on which any formula is based is a condition of equality existing between a combination of numbers arranged in a certain manner. The following example shows a simple combination of numbers of this kind:

$$2+4=5+1$$

The equality sign (=) is placed as the connecting link between the numbers 2 and 4 on one side, and the numbers 5 and 1 on the other side. An expression of this kind is known as an **equation,** because the equality sign connects two combinations of numbers that are equal, the sum of 2 and 4 being equal to the sum of 5 and 1.

This condition of equality is not limited to *sums* of numbers only, but may refer to any combination of numbers, subjected to the arithmetical operations of addition, subtraction, multiplication, division, square root, etc. The following shows an equation in which there is a combination of some of these operations:

$$(9\times8)-2=7\sqrt{64}+\tfrac{42}{3}$$

It is found that if the operations are carried out as indicated, the results on both sides of the equality sign are equal to 70.

7. Quantity.—It is convenient to have a term that may be applied to the whole expression on either side of the equality sign. For this purpose the word *quantity* is in general use. The term **quantity,** as used in mathematics, is applied to any number or combination of numbers on which the ordinary arithmetical operations are to be performed. The numerical combinations $(9\times8)-2$ and $7\sqrt{64}+\tfrac{42}{3}$, mentioned in the preceding example, may both be referred to as quantities.

When the term *quantity* is used with this meaning, an equation may be said to be an expression of equality between two quantities.

8. First and Second Members.—Distinction is made between the quantities on either side of the equality sign in an equation by applying the term *first member* to the quantity on the left-hand side of the equality sign. The quantity on the other side is called the *second member*. Speaking of both quantities, they are known as *the members of the equation*. In the following example, 5×9 is the first member and $60-15$ is the second member. Thus,

$$\overset{\textit{first member}}{5\times 9} = \overset{\textit{second member}}{60-15}$$

9. Comparison Between Equation and Scale.—Equations may be better understood by comparison with the ordinary scale, as used in stores. A scale of this kind, as shown in Fig. 1, has two pans *a* and *b*, which are supported by a lever, pivoted in the stand *c*. When the pointer *d* stands at the zero mark, it indicates that a condition of equality exists between the weights supported by the two pans; that is, the weights are in balance, or equilibrium. In the following examples the scale will be indicated diagrammatically by the balance lever, the two pans, and the weights they contain.

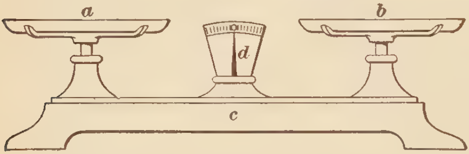


Fig. 1

10. A mathematical equation and a balance have this property in common, that a condition of equality cannot exist unless the quantities on both sides of the equality sign, or the weights in both pans of the scale, are equal.

It is impossible to reduce the weights placed in only one pan without disturbing the balance of the scale, just as it is impossible to reduce the value of only one member of the equation without disturbing the equality. But if the same change is made in both weights the scale will remain in balance, and likewise if the same change is made in both members of an equation they will remain equal.

The changes to which the members of an equation may be subjected are conveniently divided into the following four cases:

1. Adding the same quantity to both members of an equation.
2. Subtracting the same quantity from both members.
3. Multiplying both members by the same quantity.
4. Dividing both members by the same quantity.

11. Adding a Quantity to Both Members of an Equation.—It can be proved that when a number is added to one member of an equation, a number of equal value must be added to the other member if equality is to be maintained. For example, Fig. 2 (*a*) represents a scale with two pans *a* and *b*. The pan *a* supports weights of 10 and 8 pounds; and the pan *b* supports weights of 12 and 6 pounds. The total weight on each pan is 18 pounds. If, now, 1 pound is added to the pan *b*, as in Fig. 2 (*b*), the balance is destroyed, but may be regained by placing a similar weight, 1 pound, on the pan *a*, as shown in Fig. 2 (*c*).

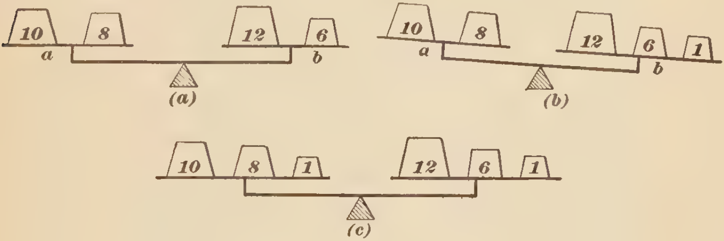


FIG. 2

The arrangement shown in Fig. 2 (*a*) may be represented by an equation, as follows:

$$10+8=12+6$$

If 1 pound is added as in Fig. 2 (*b*), the resulting change may be shown in the following manner, the vertical dash (|) being used merely to separate the members on the two sides:

$$10+8 \mid 12+6+1$$

It is seen that the equality is destroyed and that this expression is not an equation, as the value 18 of the first member is not equal to the value 19 of the second member. To regain equality, a quantity equal to 1 may be added to the left member, making the two members equal, as follows:

$$10+8+1=12+6+1$$

It follows that if a quantity is added to one member of an equation, the same quantity must be added to the other member, to maintain the equality between the members.

12. Subtracting a Quantity From Both Members.—If a number is subtracted from one member of an equation, a number of the same value must be subtracted from the other member. For example, the weight on each of the scale pans *a* and *b*, Fig. 3 (*a*), is equal to 27 pounds. If the 2-pound weight is removed from the pan *a* the balance is destroyed, but it will be regained by removing the 2-pound weight from the pan *b*, as in Fig. 3 (*b*).

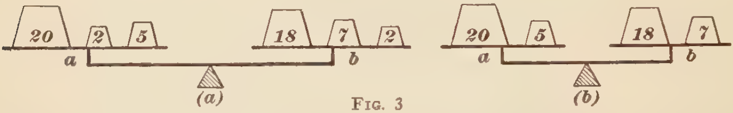


FIG. 3

The arrangement shown in Fig. 3 (*a*) is represented by the following equation:

$$20+5+2=18+7+2$$

Removing the 2-pound weight from the pan *a*, Fig. 3 (*a*), is equivalent to subtracting 2 from the first member of the equation; thus,

$$20+5+2-2 \mid 18+7+2$$

Performing the subtraction indicated in the first member, the result is $2-2=0$, and the relation will assume the following form:

$$20+5 \mid 18+7+2$$

The equality is seen to be lost, and to regain it, the same value, 2, is subtracted from the second member, as follows:

$$20+5=18+7+2-2=18+7$$

From the preceding examples the following rule is derived:

Rule.—*If a quantity is added to or subtracted from one member of an equation, the same quantity must be added to or subtracted from the other member.*

if this rule is not complied with, the equality ceases to exist; that is, the quantity represented by one member is not equal to the quantity represented by the other member.

13. Multiplying or Dividing Both Members by Same Quantity.—The remarks made about adding a quantity to, or subtracting it from, one member of an equation apply also to the processes of multiplying or dividing a member by a given number. These operations are, in reality, operations of addition or subtraction, repeated a given number of times.

For example, let it be supposed that on the scale pan *a*, Fig. 4 (*a*), there is a weight of 15 pounds, and on the pan *b* the weights of 12 pounds and 3 pounds, also equal to 15 pounds. It is required to multiply the weight on the pan *a* by 3. This is equivalent to increasing the total number of weights to three, or 15+15+15=45 pounds. The condition will then correspond to that explained in Art. **11**; that is, a weight equal to 45−15=30 pounds has been added to the pan *a*. If equilibrium is to be reestablished, the remedy is the same; that is. the weights on the pan *b* must be multiplied by the same number, or 3. These may be as shown in Fig. 4 (*b*), namely, three weights of 12 pounds each and three weights of 3 pounds each. The total weight on the pan *b* is 3×12+3×3=45 pounds.

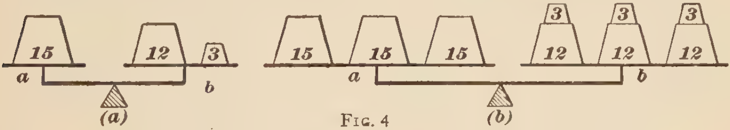


FIG. 4

When this method is applied to an equation the same results are obtained. The arrangement shown in Fig. 4 (*a*) is represented by the equation

$$15=12+3$$

When the first member is multiplied by 3, the relation assumes the following form:

$$15\times3 \mid 12+3$$

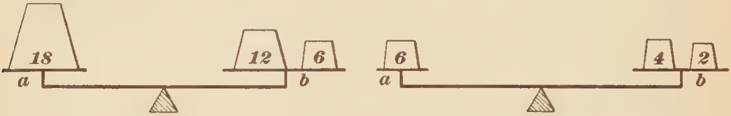
This expression shows that the equality is lost, and to regain it, the second member must also be multiplied by 3, thus:

$$15\times3=(12+3)\times3=15\times3$$

The latter equation shows the application of the parenthesis, as explained in Art. **3**. In this case, it is necessary to consider the sum 12+3 as one number, when multiplying by 3. It is therefore enclosed in parenthesis.

14. Finally, it is necessary to consider the effect produced by dividing one member of an equation by a given number. In Fig. 5 (*a*) there is seen to be a balance between the weight of 18 pounds, resting on the pan *a*; and the two weights of 12 and 6 pounds, respectively, resting on the pan *b*.

If it is required to divide the weight of 18 pounds by 3, that is, to replace it with a weight one-third as great, or 6 pounds, the evident result will be that the scale is thrown out of balance. To regain equilibrium the weights on the pan *b* must be divided by the same factor, 3. One-third of 12 and of 6 pounds is 4 and 2 pounds, respectively. When the weights on the pan *b* are replaced by these reduced weights, the arrangement will appear as in Fig. 5 (*b*), where it is seen that the pans are again in equilibrium.



(*a*) FIG. 5 (*b*)

On the application of this principle to an equation, similar results are obtained. The arrangement shown in Fig. 5 (*a*) is represented by the equation

$$18=12+6$$

When the first member is divided by 3, the relation assumes the following form:

$$18 \div 3 \mid 12+6$$

or,

$$6 \mid 12+6$$

It is seen that, as 6 cannot be equal to 18, the equality is lost, and to regain it, the second member must also be divided by 3, as in the following equation:

$$6=(12+6)\div 3=18\div 3$$

or,

$$6=6$$

Also, in this equation, it is necessary to enclose the second member of the equation, or $12+6$, in a parenthesis, before dividing by 3. Otherwise, only the last number, 6, would be divided by 3.

15. Raising a number to a given power is a process of multiplication; that is, of multiplying a number by itself a given number of times. Also, extracting the root of a power is a process of division, as the root must go into its power, and the successive quotients, as many times as the index of the root indicates. Thus, a number may be divided three times by its cube root. For example, the cube root of 27 is 3; hence, $27 \div 3 = 9$, and $9 \div 3 = 3$, and $3 \div 3 = 1$.

It follows that the rules applying to the multiplication and division of the members of an equation by a given number also apply to raising the members to like powers or extracting like roots of both members. For example, in the equation

$$6 = 4+2$$

both members may be squared without affecting the equality of the equation. Thus,

$$6^2 = (4+2)^2$$

or,

$$36 = 36$$

Likewise may like roots of both members be extracted. For example, in the equation

$$64 = 44+20$$

the square root of both members may be extracted; thus,

$$\sqrt{64} = \sqrt{44+20}$$

or,

$$8 = 8$$

16. Transformation.—From the explanations given in Arts. **11** to **15** the following important rule is derived:

Rule.—*If one member of an equation is subjected to any arithmetical operation, the other member must be treated in the same manner.*

Changes made in an equation by applying this rule are called **transformations.**

This rule, which is apparently very simple, is of the greatest importance in operations with equations. If not complied with in all cases, serious errors will result. The application of the rule is shown by the following examples:

Example 1.—It is required to add 25 to the second member of the equation

$$82+17+14=78+25+10$$

What changes must be made in the equation in order that the equality of the members may be maintained?

Solution.—By the rule, it is necessary to add a number to the first member corresponding with that added to the second one. Or,

$$82+17+14+25=78+25+10+25. \quad \text{Ans.}$$

In its present form the value of each member is 138. In its original form the value was 113; but in both cases there exists a condition of equality.

Example 2.—It is desired to subtract 34 from the first member of the following equation: $108+15+23=92+36+18$. What change must be made in the second member?

Solution.—According to the rule, if 34 is subtracted from the first member, a number of equal value must be subtracted from the second member. The equation must, therefore, be written as follows:

$$108+15+23-34=92+36+18-34$$

On subtracting 34 from a larger number in each member, the equation will be as follows:

$$74+15+23=92+2+18. \quad \text{Ans.}$$

In its present form, each member of the equation is equal to 112. In the original equation the value was 146.

Example 3.—If the second member in the equation $94+3=53+44$ is multiplied by 12, what changes must be made in the equation to maintain the equality of the members?

Solution.—According to the rule, it is not allowable to multiply the second member by 12, unless the first member is multiplied by the same factor. The equation must, therefore, be written as follows:

$$(94+3)12=(53+44)12. \quad \text{Ans.}$$

Each member must be enclosed in a parenthesis to insure that both numbers are multiplied by 12, and not only the last one.

Example 4.—The second member of the following equation is to be divided by 6. If the equality is to be maintained, what change is necessary in the equation?

$$81+17-20=64-4+18$$

Solution.—According to the rule, the equation must be written as follows:

$$(81+17-20)\div 6=(64-4+18)\div 6. \quad \text{Ans.}$$

As the whole of each member is to be divided by 6, the members must be enclosed in a parenthesis as shown.

17. Transposition.—When equations are used, it is often necessary to change a number from one member to the other; changes of this kind are called **transpositions.**

$$15+9+3=20+6+1 \qquad (1)$$

Suppose that in equation **1** it is required to move the number 20 from the right side to the left side of the equality sign. It was explained in Art. **12** that the same number may be subtracted from both members of an equation. Performing this operation by subtracting 20 from both members, the equation will appear as follows:

$$15+9+3-20=20+6+1-20$$

But, in the second member, $20-20=0$. Hence, the equation will be

$$15+9+3-20=6+1 \qquad (2)$$

An examination of equation **2** shows that the equality of the equation is not affected, as, in the first member, $15+9+3-20=7$, and, in the second member, $6+1=7$.

When the equations **1** and **2** are compared,

$$15+9+3=20+6+1$$

and

$$15+9+3-20=6+1$$

it is seen that on transferring 20 from the second to the first member its sign is changed from + to −.

18. It remains to be shown how a number, preceded by a minus sign, may be transposed from one member to the other member of an equation. For example, in the equation

$$105+15-30=40+50$$

it is desired to transfer the number −30 from the first to the second member of the equation. According to the rule in Art. **16**, it is permissible to add the same number to both members. Hence, if 30 is added to both members, the equation will appear as follows:

$$105+15-30+30=40+50+30$$

In the first member, the numbers may be written $105+15+30-30$; but $30-30=0$. The equation may, therefore, be written

$$105+15=40+50+30$$

As both members of the equation are equal to 120, the equality of the members has not been affected by the change, and the -30 has been transferred simply by changing its position and its sign.

When no sign precedes a number, a + sign is understood; thus, in the preceding equation, 105 has no sign before it, and therefore it is understood to be $+105$.

From the two preceding examples the following rule is derived:

Rule.—*A number or quantity preceded by a plus or a minus sign may be transposed from one member of an equation to the other member, if its sign is changed.*

Example 1.—In the equation $32+96+40=100+48+20$, it is required to transfer 32 to the second member.

Solution.—When no sign precedes a number, it is understood that the plus sign is omitted. Hence, the 32 in the first member is considered as $+32$, and by the rule, the equation may be written as follows:

$$96+40=100+48+20-32$$

The sum of the numbers in the first member is 136. In the second member the sum of $100+48+20=168$, and $168-32=136$. Hence, the value of each member is equal to 136.

Example 2.—Transfer the value -46 from the second to the first member in the following equation:

$$96+40+30=212-46$$

By the rule, the equation may be written as follows:

$$96+40+30+46=212$$

When the number -46 is placed in the first member its sign is changed from − to +, and each member has a value of 212.

19. Certain numbers may be transferred from one member of an equation to another by means of multiplication or division. For example, if in the equation $\frac{800}{5}=160$ it is required to transfer the denominator 5 to the second member, the rule in Art. **16** is employed and both members are multiplied by 5. Thus,

$$\frac{800\times 5}{5}=160\times 5$$

In the first member the figures 5 in the numerator and denominator cancel each other, and the equation may be written as follows:

$$800=160\times 5$$

It is seen that the equality between the members is not affected.

20. Dividing both members of an equation by the same number is also used as a means for transferring certain numbers or quantities from one member to the other. For example, in the equation

$$450=50\times 9$$

let it be assumed that 50 is to be transferred to the first member, for the purpose of having 9 alone in the second member. This result may be obtained by dividing both members by 50. Thus,

$$\frac{450}{50}=\frac{50\times 9}{50}$$

But, in the second member, the numbers 50, in the numerator and denominator, cancel each other, and the equation may now be written as follows:

$$\frac{450}{50}=9$$

21. The examples in Arts. **19** and **20** show that if the number that is to be transferred is a *denominator* of a fraction, both members must be *multiplied* by this number. If the number or factor is a *numerator*, both members must be *divided* by the number. If two or more factors are to be transferred, the equation is treated in a similar manner. In all cases the following rule applies:

Rule.—*A factor may be transferred from one member of an equation to the other by multiplying or dividing both members of the equation by the factor that is to be transferred.*

EXAMPLE 1.—In the equation $325 \times 8 = 2{,}600$ transfer the factor 8 to the second member of the equation.

SOLUTION.—The factor 8 being in the numerator of a fraction having 1 as a denominator, it follows from the preceding explanations and rule that both members must be divided by 8. Thus,

$$\frac{325 \times 8}{8} = \frac{2{,}600}{8}$$

Equal factors in the first member may now be canceled and the equation acquires the following form:

$$325 = \frac{2{,}600}{8}. \quad \text{Ans.}$$

EXAMPLE 2.—Transfer the number 5 to the second member of the following equation: $\frac{4{,}500}{5} = 900.$

SOLUTION.—By the rule, both members are multiplied by 5, giving the equation the following form:

$$\frac{4{,}500 \times 5}{5} = 900 \times 5$$

The two numbers 5 in the first member cancel each other, and the equation will be

$$4{,}500 = 900 \times 5. \quad \text{Ans.}$$

22. In the preceding examples the members of the equations consisted of one or more factors. Equations may besides several factors have single numbers connected by plus or minus signs. For example, $\frac{81}{9} = 10 + 4 - 5.$

If, in this equation, it is desired to transfer the number 9 from the first to the second member, the latter must be treated as one quantity by enclosing it in a parenthesis. The operation is then performed in accordance with the rule in Art. **21**, that is, by multiplying both members by 9. Thus,

$$\frac{81 \times 9}{9} = (10 + 4 - 5)9$$

The two 9's in the first member cancel each other and the changed form of the equation is

$$81 = (10 + 4 - 5)9$$

If, in an equation of the latter kind, the factor that is outside the parenthesis is transferred to the other member, there is no further use for the parenthesis, and it may be removed.

The following example will make this clear: In the equation $4\times 27=6(25-10+3)$, the factor 6 is to be transferred to the first member. Following the rule in Art. **21**, both members are divided by 6. Thus,

$$\frac{4\times 27}{6}=\frac{6(25-10+3)}{6}$$

The factors 6 in the second member cancel each other and the equation may be written

$$\frac{4\times 27}{6}=(25-10+3)$$

As the parenthesis is no longer required, it is removed and the final form of the equation is

$$\frac{4\times 27}{6}=25-10+3$$

When the operations indicated by the signs are performed, the equation is

$$\frac{108}{6}=25-10+3$$

or,

$$18=18$$

It is seen that the equality of the two members has not been affected by the change.

EXAMPLE.—In the equation $53\times 8=4(190-92+8)$ the factor 53 is to be transferred to the second member, so that the first member contains but the one factor 8.

SOLUTION.—By the rule in Art. **21**, both members are divided by 53.

Thus,

$$\frac{53\times 8}{53}=\frac{4(190-92+8)}{53}$$

The two equal factors 53 in the first member cancel and the equation appears as follows:

$$8=\frac{4(190-92+8)}{53}$$

That the members still have equal values may be ascertained by performing the required arithmetical operations. Thus, after adding and subtracting the numbers in the parenthesis, the result is as shown in the following equation:

$$8=\frac{4\times 106}{53}$$

or,

$$8=8$$

EQUATIONS FORMED WITH SYMBOLS

APPLICATION OF SYMBOLS

23. Numbers Represented by Symbols.—The preceding articles explain the fundamental properties of equations by means of numbers, so as to show more clearly the results of the various operations. In the succeeding pages symbols, such as letters, will be used to represent the various numbers used in the equations. The advantage to be derived from this method will be shown gradually, as the scope of the examples becomes more extended. No change in the method of treating equations is involved by substituting letters for numbers.

Using letters to represent numbers is not a method limited solely to equations. It is used, for example, to a great extent in stores as a means for indicating prices on articles, so as to prevent the customer from reading the prices. But, the salesman, who is furnished with a key, knows the meaning of each letter, and is therefore able to tell the price.

24. Omitting Multiplication and Division Signs.—It was stated in Art. **3** that in formulas and equations the various arithmetical operations are indicated by the same signs as were used in arithmetic. In formulas and equations, however, where brevity is always desired, it is possible in many cases to omit the multiplication and division signs, when such omissions will not cause any misunderstanding. Thus, instead of writing $a \times b$, the sign of multiplication is omitted, and the product of the two letters is indicated by writing them close together, without any intervening sign. Hence, $a \times b$ is written ab; $d \times e \times f$ is written def; $b \times (c+e)$ is written $b\,(c+e)$, and so forth. This simplified method of indicating multiplication cannot, however, be adopted for numbers. For example, 3×7 cannot be written 37, for 37 has been defined, in arithmetic, to mean $30+7$.

In connection with the subject of powers, it follows that, as $3 \times 3 = 3^2$, so $a \times a = a^2$. This latter expression is read *a square*, following the rule given in *Powers and Roots.* If there are three equal factors, as $b \times b \times b$, the result is b^3, which is read *b cube;* b^4 is read *b fourth* and is equal to $b \times b \times b \times b$; c^5 is *c fifth* and is equal to $c \times c \times c \times c \times c$; and so on for any other power.

The sign of division ($\div$) is not so generally used to indicate division of symbols as it is to indicate division of numbers. Often the dividend is written as the numerator of a fraction of which the divisor is the denominator; thus, $a \div b$ may be expressed as $\frac{a}{b}$ just as $1 \div 2$ may be written $\frac{1}{2}$ or $4 \div 3$ as $\frac{4}{3}$.

25. Supplementary Symbols.—Only a limited number of letters are employed in the equations dealt with in the succeeding pages. In practice, the letters of the alphabet are at times, insufficient to meet the requirements, and it is found necessary to have recourse to supplementary marks and symbols, known as *primes* and *subscripts*. The marks $'$, $''$, $'''$, are called **primes;** small letters or numbers, as $_a$, $_b$, $_c$, etc. and $_1$, $_2$, $_3$, etc., printed below the line, are known as **subscripts,** or **subs.**

When capital letters have primes annexed to them, as A', B', A'', C''', they are commonly read *large a prime*, *large b prime*, *large a second*, *large c third*, respectively; but when small letters are used, as a', b', c'', d''', they are read *small a prime*, *small b prime*, *small c second*, *small d third*, respectively. Similarly, such expressions as a_1, D_2, W_3, f_4 are read *small a sub one*, *large d sub two*, *large w sub three*, *small f sub four*, respectively.

The words *large* and *small* are used in reading these expressions only when both capitals *and* small letters occur in the same problem. If all the letters are capitals, or if all the letters are small letters, the words *large* and *small* are dropped. Under these conditions, F', P_2, R'' would be read *f prime*, *p sub two*, *r second;* and d_1, g'', a_3 would be read *d sub one*, *g second*, *a sub three.*

In writing the different letters, the capital letters are formed like *printed* letters, so that they may be easily distinguished from small letters. On the whole, care should always be taken in writing letters to see that there is no confusion; that, for instance, subs are not mistaken for exponents, and that each letter represents just what is intended.

OPERATIONS WITH THE TERMS OF AN EQUATION

26. Rules Applying to Equations.—All the rules for equations formed with numbers apply also to equations in which letters are used and will, therefore, require no further explanations. The application of the preceding rules will be shown by means of examples, and if it is kept in mind that the letters stand for certain given numbers, there should be no difficulty in applying the rules.

The following example will illustrate the application of the rule, in Art. 12, stating that no quantity can be added to or subtracted from one member of an equation without performing a similar operation with the other member. The given equation is:

$$a+b+c=d+e+f$$

Supposing that the symbol g is to be *added* to the first member, it must also be added to the second member, and the changed equation will assume the following form:

$$a+b+c+g=d+e+f+g$$

These letters may have any value, but the arrangement of the letters in the two members must be such as to comply with the law that the two members must have equal values. In the present case the values of the letters are assumed to be as follows:

$a=1$
$b=3$
$c=9$
$d=2$
$e=4$
$f=7$
$g=8$

When these values are substituted for the letters, the equation assumes the following form:

$$1+3+9+8=2+4+7+8$$

or,

$$21=21$$

If the letter g is *subtracted* from the first member, it must according to the rule, be subtracted also from the second member. Thus,

$$a+b+c-g=d+e+f-g$$

The substitution of the given values makes the equation

$$1+3+9-8=2+4+7-8$$

or,

$$5=5$$

27. The rule in Art. **16**, which shows that both members of an equation may be multiplied or divided by the same number, is illustrated by means of the following examples:

Example 1.—The first member of the equation $a=b+c$ is to be multiplied by the factor d.

Solution.—According to the rule both members must be multiplied by d, or

$$a\,d=(b+c)d. \quad \text{Ans.}$$

Example 2.—In the equation $a=b+c$, the second member must be divided by d.

Solution.—To comply with the rule both members must be divided by d; hence,

$$\frac{a}{d}=\frac{b+c}{d}. \quad \text{Ans.}$$

Proof.—In examples 1 and 2 the letters have the following values:

$$a=12,\ b=9,\ c=3, \text{ and } d=5$$

The substitution of these values in example 1 makes the given equation $12=9+3$, and the solution is $12\times5=(9+3)\,5$. In example 2 the solution is

$$\frac{12}{5}=\frac{9+3}{5}$$

28. Expressions and Terms.—Any symbol, or combination of symbols, used to represent a mathematical quantity is called a **mathematical expression,** or simply an **expression.** Thus, 24 and a, each representing a number, are expressions. Combinations of numbers and letters or of several letters, such as $13a$, $a-b$, ab, a^2+b^2, $\frac{ab}{c}$, and $2+5a$, are all mathematical expressions, because each may represent a number.

In the foregoing expressions addition, subtraction, multiplication, and division are indicated. In many cases it is convenient to have a special name for parts of an expression.

An expression, or a part of an expression, which is not separated into parts by a plus or a minus sign is called a **term.**

The expression $c\,e-c^2 r$ may then be said to consist of the two terms $c\,e$ and $c^2\,r$, connected by a minus sign; the expression $a+b-c$ consists of the three terms a, b, and c.

29. Coefficients and Literals.—When a term is expressed by letters and a number, written before the letters, the number is called the **coefficient.** The coefficient always precedes the letters of the term; thus in $8\,a\,b$ the figure 8 is the coefficient. The coefficient shows how many times the rest of the term is to be taken. In the term $8\,a\,b$ the coefficient 8 indicates that the quantity $a\,b$ is to be taken eight times.

At times it is convenient to have a special name also for the portion of a term that consists of letters. This part of the term is called the **literal expression, literal quantity,** or simply the **literal.**

When no coefficient is written in a term, the coefficient 1 is always to be understood. Thus, a means $1\,a$, and $c\,d$ means $1\,c\,d$.

30. Like Terms.—If the sum of a number of similar coins is to be found they are simply counted and the number found is the sum. Thus, counting six single cents, the sum is written 6 cents. Similarly, if the letter c represents 1 cent, the expression $c+c+c+c+c+c$ means that the number of c's is to be found by counting, or addition. The fact that there are six c's is represented by the expression $6\,c$, the coefficient 6 indicating the number of c's.

In counting money, unlike coins are kept separate, as one could not count a combination of cents and dimes, as representing either cents or dimes. Similarly, in the expression $c+c+c+d+d+d$, one could not add all the letters and state the sum as $6\,c$ or $6\,d$, as both would be wrong. Hence, similar letters must be grouped together and then added. In the preceding example, the sum is $3\,c+3\,d$.

31. Terms that contain the same letters or combination of letters are called **like terms;** all others are **unlike terms.** The term *like* does not mean, in this case, that the terms are of equal *values*, but that the terms are of the same *kind*. Thus, the terms $3\,c$ and $5\,c$ are like terms, because both terms contain the same literals c, but they are not of the same value, as the coefficients 3 and 5 are not equal. Similarly, the expressions 3 cents and 5 cents are like, as both deal with cents, but the values of the two expressions differ.

In the terms $3\,a\,b$, $4\,a\,b$, and $9\,a\,b$ the literals, $a\,b$, are the same in all of them; hence, they are like terms. If the letters have the same exponents, they are like terms. Thus, the terms $3\,c\,d^2$, $4\,c\,d^2$, and $6\,c\,d^2$ are like terms, as not alone the letters, but also the respective exponents, are alike. It is important that this subject of *like* terms be well understood, so as to avoid serious mistakes in addition and subtraction of terms. Remember that *like terms are those which have the same letters affected by the same exponents, though the terms need not have the same coefficients.*

32. Unlike Terms.—Terms are **unlike** if they contain different combinations of letters, or if the exponents of the letters are not alike. For example, the terms $2\,a$, $13\,b$, and $27\,c$ are unlike terms, because they contain different letters. Supposing that a, b, and c represent hours, minutes, and seconds, respectively, it is obvious that the sum of the terms $2\,a$, $13\,b$, and $27\,c$ cannot be found by adding the coefficients 2, 13, and 27, as the sum, 42, would not represent either hours, minutes, or seconds. Hence, in adding these terms, the sum must be written $2\,a+13\,b+27\,c$, just as one would write 2 hours 13 minutes 27 seconds; the plus signs are omitted in the latter case but addition is understood.

The terms $2\,a\,b$, $5\,a\,c$, and $3\,b\,c$ are unlike terms, as each one contains a different combination of letters. The terms $2\,a^2\,b$, $2\,a^3\,b$, $3\,a\,b$, $7\,a\,b^2$, $4\,a\,b^3$, and $5\,a^2\,b^2$ are all unlike terms, notwithstanding the fact that the terms are composed of the same letters. They are unlike terms, because the exponents of similar letters are different in each term.

33. Addition of Like Terms.—In a term consisting of a coefficient and one or more letters, as $4\,a\,b$, the coefficient, 4, indicates how many times the quantity $a\,b$ is to be taken; in this case four times, or $ab+ab+ab+ab$. Let it be supposed that $a=2$, and $b=3$; the expression ab is then equal to $2\times 3=6$, and the expression $ab+ab+ab+ab$ would be equal to $6+6+6+6$. The term $4ab$ would then be equivalent to four sixes. For similar reasons the term $3ab$ would be equal to three sixes. The expression $4ab+3ab$ would, therefore, be equal to four sixes plus three sixes, or seven sixes. Hence, $4ab+3ab$ may be written $7ab$. Similarly, $5cd+2cd+14cd$ is equal to $21cd$. The principle employed in the preceding explanations may be stated as a rule, as follows:

Rule.—*To add like terms, add the coefficients, and to the sum annex the common literal part.*

34. Subtraction of Like Terms.—An illustration similar to the one given in the preceding article may be used to show that the difference between two like terms may be found. The difference between $12cd$ and $9cd$ is $12cd-9cd$ or $3cd$, just as 7 inches $-$ 4 inches is 3 inches. Here the difference between the coefficients is found and the common letters are annexed. The rule for subtracting like terms is:

Rule.—*To subtract like terms, find the difference between the coefficients, and to the difference annex the common literal part.*

35. In examples in subtraction it sometimes happens that the subtrahend is larger than the minuend. For instance, in the expression $5-14$, the subtrahend 14 is larger than the minuend 5 from which it is to be subtracted. Only 5 of the 14 units to be subtracted can be taken from 5 and 9 still remains not subtracted. To indicate that part of the subtrahend is still to be subtracted a minus sign is written before 9. Thus, $5-14=-9$. The minus sign shows that 9 is still to be subtracted.

If the numbers to be subtracted are represented by letters the minus sign is used similarly. If $20b$ is to be subtracted from $14b$, $6b$ will remain unsubtracted. Thus, $14b-20b=-6b$. Here the difference between $20b$ and $14b$ is found and the minus sign is written before it.

Rule.—*When a subtrahend is larger than the minuend, subtract the smaller number from the larger and place a minus sign in front of the result.*

A term that has a minus sign written before it is often called a **negative term** to distinguish it from a term preceded by a plus sign, which is called a **positive term.** If a term has no sign written before it, it is a positive term since the plus sign is understood. Thus, $2\,a\,c$ in the expression $2\,a\,c-3\,b\,d$ is a positive term.

36. If there are several like terms to be combined all the terms may be positive or all may be negative or some may be positive and some negative.

If all the terms are positive, use the rule of Art. **33.** Thus,

$$6\,ac+7\,ac+3\,ac+ac=17\,ac$$

If all the terms are negative, combine them as if they were all positive and write a minus sign before the sum. The sum of $-4\,a$, $-6\,a$, $-2\,a$, and $-a$ is $-13\,a$. The sum is found by first getting the sum of $4\,a+6\,a+2\,a+a$, which is $13\,a$, and then prefixing a minus sign to the result.

If some of the terms are positive and some negative, find the sum of the positive terms and the sum of the negative terms separately. Then combine the two sums by using the rule of Art. **34** or of Art. **35.**

An expression which is not made up entirely of like terms can often be simplified by combining the terms which are like. Thus, to simplify $x^2+y+3\,x^2-2\,x^2-2\,y$, add the terms containing x^2 and those containing y separately and write the results, preceded by their signs, as the terms of an expression. First find the sum of the terms containing x^2.

$$x^2+3\,x^2-2\,x^2=4\,x^2-2\,x^2=2\,x^2$$

The sum of the terms containing y is $y-2\,y$ or $-y$. Then,

$$x^2+y+3\,x^2-2\,x^2-2\,y=2\,x^2-y$$

EXAMPLE 1.—Find the sum of $2\,x^2y+4\,x^2y+3\,x^2y$.

SOLUTION.—The terms are all like terms and all positive. Therefore, apply the rule of Art. **33.** The sum of the coefficients 2, 4, and 3 is 9.

The common literal part is x^2y, which is annexed to 9. Thus, the sum of $2\,x^2y+4\,x^2y+3\,x^2y$ is $9\,x^2y$. Ans.

EXAMPLE 2.—Find the sum of $-6a^2$, $-3a^2$, $-11a^2$, and $-a^2$.

SOLUTION.—All the terms are like terms and all are negative. Therefore, find the sum of the terms as if all were positive and write a minus sign before the result.

$$6\,a^2+3\,a^2+11\,a^2+a^2=21\,a^2$$

Then, $$-6\,a^2-3\,a^2-11\,a^2-a^2=-21\,a^2. \quad \text{Ans.}$$

EXAMPLE 3.—Combine the terms in the following expression:

$$2\,b-4\,b-6\,b+8\,b+3\,b$$

SOLUTION.—First combine the positive terms; then combine the negative terms.

$$2\,b+8\,b+3\,b=13\,b$$
$$-4\,b-6\,b=-10\,b$$

The last step is to combine $13\,b$ and $-10\,b$. According to Art. **34**, $13\,b-10\,b=3\,b$. Ans.

EXAMPLE 4.—Combine like terms in the following expression:

$$14\,a\,b+3\,a\,b-15\,a\,b-9\,b\,c+14\,b\,c-12\,b\,c$$

SOLUTION.—When the like terms having the same sign are combined, the expression becomes $17\,a\,b-15\,a\,b+14\,b\,c-21\,b\,c$; $17\,a\,b-15\,a\,b=2\,a\,b$ and $14\,b\,c-21\,b\,c=-7\,b\,c$. The result is $2\,a\,b-7\,b\,c$. Ans.

The rules for combining like terms are applied when the coefficients are either fractions or decimals; thus $\frac{3}{4}\,b+\frac{1}{2}\,b=\frac{5}{4}\,b$. Here the sum of the coefficients $\frac{3}{4}$ and $\frac{1}{2}$ is $\frac{5}{4}$. Similarly $A+.45\,A=1.45A$. Since the coefficient of A is understood to be 1, the sum of the coefficients is $1+.45$ or 1.45.

37. Application of Rules.—The preceding rules apply to any expressions irrespective of whether they stand alone or form parts of an equation. In the following examples the various expressions are connected by an equality sign; they represent, therefore, equations and the rules in Art. **18** must be applied when a term is transposed from one member of the equation to the other member.

EXAMPLE 1.—Add like terms in the equation $4\,a+2\,b+3\,a=8\,c$.

SOLUTION.—The terms $4\,a$ and $3\,a$ being like terms, they may be added by adding the coefficients; thus, $4\,a+3\,a=7\,a$. The equation may now be written as follows: $7\,a+2\,b=8\,c$. Ans.

EXAMPLE 2.—Combine the like terms in the equation $16\,a+4\,b+9\,c=8\,a+2\,d+7\,c$.

SOLUTION.—It is necessary to have the terms that are to be combined situated in the same member; according to the rule in Art. **18**, a term may be transposed from one member of the equation to the other by changing the sign of the term. Hence, $8a$ and $7c$ may be transposed to the first member, where they become $-8a$ and $-7c$; then the equation is written as follows:

$$16a-8a+4b+9c-7c=2d$$

But $16a-8a=8a$, and $9c-7c=2c$. Hence, the equation is now

$$8a+4b+2c=2d. \quad \text{Ans.}$$

EXAMPLE 3.—Combine like terms in the equation $5a+\frac{1}{2}a+4b-9b=16c-1\frac{1}{2}c$.

SOLUTION.—In the first member of the equation the sum of $5a$ and $\frac{1}{2}a$ is $5\frac{1}{2}a$, and $+4b-9b=-5b$, according to the rule in Art. **35**; in the second member $16c-1\frac{1}{2}c=14\frac{1}{2}c$. The simplified equation may now be written:

$$5\tfrac{1}{2}a-5b=14\tfrac{1}{2}c. \quad \text{Ans.}$$

EXAMPLE 4.—Combine like terms in the following equation:

$$9.13a-16.42b=11.47c-2.93b-5.25a+8.32a$$

SOLUTION.—When like terms are transposed to the same side of the equation, the equation may be written as follows:

$$9.13a+5.25a-8.32a=16.42b-2.93b+11.47c$$

NOTE.—At the present stage it does not matter whether the terms are transposed to the first or to the second member; but, when the conditions are such as to require one term to be alone in one member, then the remaining terms must all be transposed to the other member.

In the equation the addition and subtraction of like terms may be performed in the following manner: $9.13a+5.25a=14.38a$; $14.38a-8.32a=6.06a$. The difference between $16.42b$ and $2.93b$ is $13.49b$.

The equation in its simplified form is:

$$6.06a=13.49b+11.47c. \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

Perform the operations indicated in the following expressions:

1. $3c+4c+7c+8c$. Ans. $22c$
2. $ab^2+3ab^2+11ab^2+13ab^2$. Ans. $28ab^2$
3. $7a^2+a^2+13a^2-5a^2$. Ans. $16a^2$

Simplify the following equations:

4. $7a+2a-3a=8b-3b$. Ans. $6a=5b$
5. $9x+4y-3z-6x-6y+4z=14-3$. Ans. $3x-2y+z=11$
6. $9xy+2yz-4xy=15xz-6xz+2yz$. Ans. $5xy=9xz$

38. Multiplication of Terms.—It has been emphasized that unlike terms cannot be added; but either like or unlike terms can be multiplied. For example, if 6 is multiplied by 3, 6 is to be taken three times, or $6+6+6$. Similarly, if a is multiplied by b, the factor a is to be taken b times, and b may here be of any value. In the first case the product is $6\times 3=18$; in the latter case it is $a\times b=a\,b$.

If a term, consisting of a coefficient and one or more letters, as $24\,a\,b$, is to be multiplied by a number, as 4, the product is found as follows: $24\,a\,b\times 4=96\,a\,b$. That is, the coefficient is multiplied by the number in the usual way and the letters are annexed to the product. If two terms containing literals, or letters, are to be multiplied together, as in the example $14\,a\,b\times 3\,c$, the numbers, or coefficients, are multiplied as before, and each letter that occurs in either term is annexed to the product. The exponent of each letter is the sum of its exponents in the terms multiplied together. Thus, the product of $14\,a\,b$ and $3\,a\,c$ is $42\,a^2\,b\,c$. First 14 is multiplied by 3; the product is 42. Then the letters a, b, and c are annexed to 42. The exponent of a will be $1+1$ or 2. The exponent of each of the other letters is 1 and is not expressed.

It is customary to arrange letters and quantities alphabetically in all algebraic expressions; thus, $4\,a\,b\,c$ rather than $4\,c\,a\,b$ or $4\,b\,a\,c$, etc. Also, write $4\,a\,b+3\,a\,c+4\,a\,d$ rather than place the term containing b after one containing c or d.

If several terms are to be multiplied together the first term is multiplied by the second. This product is multiplied by the third, and so on. Thus, $11\,a\,c\times 3\,c\times 4\,b=132\,a\,b\,c^2$. First $11\,a\,c$ is multiplied by $3\,c$. The product is $33\,a\,c^2$. Then $33\,a\,c^2$ is multiplied by $4\,b$.

39. Division of Terms.—The operation of division may be performed with either like or unlike terms. For example,

$$16\,a\,b\div c\,d=\frac{16\,a\,b}{c\,d}$$

$$(2\,a^2\,b+3\,c\,d^2)\div 5\,e=\frac{2\,a^2\,b+3\,c\,d^2}{5\,e}$$

Cancelation should be used whenever possible to simplify expressions. A few examples will show the general method to be followed. The expression $2\,a \div 2$ is equal to $\frac{2\,a}{2}$, in which the number 2 is canceled so that only the letter a remains. Hence, $2\,a \div 2 = a$. Also $6\,a \div 2\,a$ may be written $\frac{6\,a}{2\,a}$ or $\frac{6 \times a}{2 \times a}$. The letter a is canceled and 2 is canceled into 6. Thus, $\frac{6 \times a}{2 \times a} = 3$. Similarly,

$$6\,a\,b \div 2\,a = \frac{6\,a\,b}{2\,a} = \frac{6 \times a \times b}{2 \times a} = 3\,b$$

If a letter which occurs in both the dividend and the divisor does not have the same exponent in both terms it is canceled and rewritten in the term in which it had the larger exponent. The new exponent is equal to the difference between its exponent in the dividend and its exponent in the divisor. Thus,

$$8\,a^3\,b \div 4\,a = \frac{8\,a^3\,b}{4\,a} \text{ or } 2\,a^2\,b$$

Here 4 is canceled into 8. The exponent of a in the dividend is greater than its exponent in the divisor, so when a is canceled it is rewritten in the dividend with the exponent $3-1$ or 2.

EXAMPLES FOR PRACTICE

1.	Multiply $2\,x$ by $3\,y$.	Ans. $6\,x\,y$
2.	Multiply $2\,a\,c$ by $4\,a\,b$.	Ans. $8\,a^2\,b\,c$
3.	Multiply $8\,a\,c$ by $5\,b\,d$.	Ans. $40\,a\,b\,c\,d$
4.	Multiply $3\,a^2\,c$ by $4\,b^2\,c^2$.	Ans. $12\,a^2\,b^2\,c^3$
5.	Divide $2\,x$ by $3\,y$.	Ans. $\frac{2\,x}{3\,y}$
6.	Divide $12\,a\,b$ by $4\,b$.	Ans. $3\,a$
7.	Divide $9\,a^2\,c$ by $3\,a\,d$.	Ans. $\frac{3\,a\,c}{d}$
8.	Multiply $3\,x$ by $6\,y$ and divide the product by $3\,z$.	Ans. $\frac{6\,x\,y}{z}$

PARENTHESIS

40. Application to Symbols.—The use of a parenthesis, brackets, and brace with numbers is explained in Art. 3. The rules given apply also to quantities indicated by letters. The expression $(a+b)\times c$, or $(a+b)c$, may be interpreted in two different ways. Thus, $(a+b)c$ may be said to mean either that the sum of a and b is to be multiplied by c or that each term within the parenthesis is to be multiplied by c and the products added. The value of the result is the same in either case as may be seen from the following example:

If $a=4$, $b=5$, and $c=3$, then $(a+b)c=(4+5)\times 3$ or 9×3, which is 27. If the expression is read in the other way, then $(a+b)c=(4+5)\times 3=4\times 3+5\times 3=12+15$ or 27. The expression $(a-b)c$ also may be interpreted in two ways as, for instance, that b is to be subtracted from a and the remainder multiplied by c or that a and b are each to be multiplied by c and the last product subtracted from the first.

41. Removal of Parentheses.—It has just been shown how a parenthesis is removed when it encloses a factor of a term. A quantity in a parenthesis may make up a whole term of an expression. Then only addition or subtraction is indicated and the parenthesis must be removed in such a way that the indicated operations of addition or subtraction will be performed. If the numbers 42, 10, and 8 are added, the sum is 60; that is, $42+10+8=60$. If the sum of 10 and 8 is added to 42, the sum is also 60; that is, $42+(10+8)=60$. Hence, $42+(10+8)=42+10+8$. In the same way, $a+(b+c)=a+b+c$.

Again, if 42 and 10 are added together and 8 is subtracted from the sum, the result is 44; that is, $42+10-8=44$. If 8 is subtracted from 10 and the remainder is added to 42, the result is also 44; that is, $42+(10-8)=44$. Hence, $42+(10-8)=42+10-8$. In a similar manner, $a+(b-c)=a+b-c$.

From the preceding examples it is seen that, if a parenthesis is preceded by a *plus sign*, addition is indicated and the parenthesis may be removed without changing the value of the expression. From this the following rule is formulated:

Rule.—*If an expression, indicating addition and subtraction only, contains a parenthesis preceded by a plus sign, the parenthesis may be removed without changing any signs.*

42. If the parenthesis is preceded by a minus sign subtraction is indicated. For example, in the expression $42-(10+8)$ the sum of 10 and 8 is to be subtracted from 42. The result is 24. That is, $42-(10+8)=24$. Compare the result obtained with the value of the expression $42-10-8$, where 10 is subtracted from 42 and then 8 is taken from the remainder. This result also is 24. Hence, $42-(10+8)=42-10-8$. For similar reasons,

$$a-(b+c)=a-b-c$$

Again, if the remainder obtained by subtracting 8 from 10 is subtracted from 42 the result is 40. That is, $42-(10-8)=40$. If 10 is subtracted from 42 and 8 added to the difference the result is also 40. That is, $42-10+8=40$. Hence, $42-(10-8)=42-10+8$. Similarly,

$$a-(b-c)=a-b+c$$

From these examples the following rule is derived:

Rule.—*If an expression, indicating addition and subtraction only, contains a parenthesis preceded by a minus sign, the parenthesis may be removed, if the sign of every term within the parenthesis is changed.*

EXAMPLE.—Remove the parentheses from the expression $(a+b-2c)-(a-b-4c)$ and perform the required operations of addition and subtraction.

SOLUTION.—As any quantity is supposed to be preceded by a plus sign, unless another sign is indicated, the first parenthesis is supposed to have a plus sign in front of it. The parenthesis may, therefore, according to the rule in Art. **41**, be removed without changing the signs of the enclosed terms. The second parenthesis is preceded by a minus sign; hence, on removing the parenthesis, the signs of the enclosed terms must, according to the last rule, be changed. The first term a in the second parenthesis has a plus sign, understood, and when the parentheses are removed the two quantities will appear as follows: $a+b-2c-a+b+4c$.

It is to be noted that the second letter a now has a minus sign, instead of its original plus sign, and that the terms b and $4c$ have plus instead of minus signs.

The like terms may now be combined. Thus, $a-a=0$; $b+b=2b$, and $4c-2c=2c$. The simplified expression will have the following form:

$$2b+2c. \quad \text{Ans.}$$

43. The parenthesis *cannot* be removed according to the rules of Arts. **41** and **42** if the expression involves multiplication, unless each of the terms within the parenthesis is first multiplied by the factor outside the parenthesis, as explained in Art. **40.** For example, take the expression $(8+7)\times 4$. This means that the sum of 8 and 7, or 15, is to be multiplied by 4, and the result is 60; also, $4\times 8+4\times 7=60$. Now, if the parenthesis were dropped without performing the multiplication, the expression would become $8+7\times 4$, which is equivalent to $8+28=36$, showing that the dropping of the parenthesis without performing the multiplication indicated alters the value of the expression and therefore is incorrect. In other words, the parenthesis cannot be omitted from such an expression as $(a+b)c$ unless both a and b be multiplied by c. Similarly, the parenthesis cannot be omitted from the expression $160-4\times(36-28)$. As it stands, this expression indicates $160-4\times 8=160-32=128$. But if the parenthesis were omitted, and the sign changed, the equation would become $160-4\times 36+28=160-144+28=44$, which would be altogether wrong.

Therefore, the rules in Arts. **41** and **42** must not be applied if the expression in parenthesis is multiplied by any term. The rules are applicable only to cases such as have been illustrated in Arts. **41** and **42.**

EXAMPLES FOR PRACTICE

Remove the parentheses from the following expressions and combine like terms:

1. $(a+b-2c)-(a-b-c)$. Ans. $2b-c$
2. $(3e+4f-5)+(2e-5f+1)$. Ans. $5e-f-4$
3. $(2a+b)-(a-b)+(a-2b)$. Ans. $2a$
4. $(14ab-9ab)+(9b-12b)-(13bc-7bc)$. Ans. $5ab-3b-6bc$

EQUATIONS WITH ONE UNKNOWN TERM

44. Preliminaries.—In the preceding pages of this Section the elements of equations have been described in detail, and also the means by which these elements may be combined or transposed to the different parts of the equation. In the equations that have been dealt with hitherto, the terms had known values, and the advantages of using equations could, therefore, not be displayed. But, since the necessary information by which to manipulate equations has been obtained, it is now possible to take a step farther and to describe their application.

45. Elementary Equations.—The fundamental parts of an equation are the two members connected by an equality sign, as $2+3=5$.

If one of the terms of an equation is removed or covered, the required conditions of equality will be a means for ascertaining what its value must be. Thus, suppose the black circle in the following equation represents a coin covering one of the terms of the equation

$$3+5+\bullet=10+7$$

The fundamental requirement of an equation is that both members must be of equal value; therefore, the value of the first member must be equal to that of the second member, or $10+7=17$. The sum of 3 and 5 in the first member is 8, and the number covered must therefore be $17-8=9$. The complete equation is

$$3+5+9=10+7$$

46. In the following equation the second term is hidden and its value is to be found:

$$42-\bullet=18\times3-30$$

The value of the second member is equal to $54-30=24$. As the first member must have the same value, the hidden term must be $42-24=18$, and the complete equation is

$$42-18=18\times3-30$$

In the following equation the denominator of the first member is hidden, and must be found:

$$\frac{8\times4}{\bullet}=56-30-10$$

The value of the second member is $56-40=16$, and the unknown number must be such that when divided into 32 the quotient will be 16; that is, the product of the unknown number and 16 is 32. Therefore, the unknown number must be $32\div16$, or 2, and the complete equation is

$$\frac{8\times4}{2}=56-30-10$$

or,

$$16=16$$

47. Symbol for Unknown Term.—In all equations it is customary to indicate the position of the unknown term by means of a letter, and for this purpose the letter x is in general use. Thus, if the sum is to be found of the numbers 5, 3, 9, and 2, they may be arranged to form an equation by letting the letter x represent the unknown sum. Or,

$$x=5+3+9+2$$

On performing the addition, the sum is found to be 19. Hence,

$$x=19$$

In this example the unknown term x constitutes the *first* member of the equation and the known terms the *second* member. This is the most convenient arrangement for finding the value of an unknown term x, and the various transformations to which an equation is subjected aim to separate x from the other terms.

If x is used for the hidden or unknown term in the first equation in Art. 46, the equation is

$$42-x=18\times3-30$$

Then, as has been shown,

$$x=18$$

The next equation would be

$$\frac{8\times4}{x}=56-30-10$$

if x were written for the hidden term. Here $x=2$.

48. Solving an Equation.—An equation is said to be solved when the value of the unknown quantity is found. The unknown quantity will then make up one member of the equation.

An equation that is to be solved may be stated so that some of the terms containing the unknown quantity are in one member and some in the other; then, the first step in solving is to transpose so that all the unknown terms are in the first member and all the others are in the second. The rule to be used is given in Art. 18. Next, the like terms are combined; then both members of the equation may be divided by the coefficient of x. For example, to solve the equation

$$20+5x-18=10+3x$$

the first step is to transpose the terms so that $5x$ and $3x$ will make up the first member and 10, 20, and 18 will be in the second member. Since the sign before $3x$ is plus, the sign will become minus when $3x$ is written in the first member; similarly, the sign of 20 will become minus and the sign of 18 plus when these terms are written in the second member. After the terms are transposed the equation is

$$5x-3x=10-20+18$$

The terms in each member are combined.

$$5x-3x=2x$$

and $$10-20+18 \text{ or } 10+18-20=8$$

Then, $$2x=8$$

When the first member is divided by 2, the coefficient of x, it becomes $\frac{2x}{2}$ or x; thus, by performing this division, x may be made the first member. If the first member is divided by 2, the second member must also be divided by 2; the equation then becomes

$$x=4. \quad \text{Ans.}$$

49. Clearing an Equation of Fractions.—When an equation contains a fraction in which the unknown quantity is one of the terms, it must be changed so that the fraction does not appear. This process is called clearing an equation of fractions and since it is most important it must be thoroughly understood. The principle on which the process is based will be explained by means of an example. Suppose, for instance, that the value of x is to be found in the equation

$$\frac{x}{2}+\frac{x}{3}=25$$

It has been explained that both members of an equation may be multiplied by the same number without destroying the equality of the members. If both members of this equation are multiplied by a number that will contain each denominator without a remainder, as 6, 12, or 18, the denominators can be canceled. If 6 is used as a multiplier, the equation becomes

$$6\times\frac{x}{2}+6\times\frac{x}{3}=6\times 25$$

or,

$$3x+2x=150$$
$$5x=150$$
$$x=\ 30. \quad \text{Ans.}$$

In this example, 6 is the least common denominator of $\frac{x}{2}$ and $\frac{x}{3}$ and in clearing an equation of fractions the least number that can be used as a multiplier is the least common denominator of all the fractions in the equation. The method of finding the least common denominator is explained in a preceding Section.

Rule.—*Any equation can be cleared of fractions by multiplying both members by the least common denominator of the fractions.*

Example 1.—Find the value of x in the equation $\frac{2\,x}{3}+\frac{x}{5}=45$.

Solution.—According to the rule, both members are multiplied by the least common denominator, which is equal to $3\times 5=15$. Thus,

$$\frac{15\times 2\,x}{3}+\frac{15\,x}{5}=15\times 45$$

The denominators may now be canceled by dividing the numerators by 3 and 5, respectively; thus,

$$5\times 2\,x+3\,x=15\times 45$$

or

$$10\,x+3\,x=15\times 45$$

$$13\,x=15\times 45$$

$$x=\frac{15\times 45}{13}=51\tfrac{12}{13}. \quad \text{Ans.}$$

EXAMPLE 2.—Solve the equation $2\,x+\frac{x}{5}=\frac{2}{3}+\frac{3}{5}$ to find the value of x.

SOLUTION.—The equation is multiplied by the least common denominator, which is 15. Thus, $15\times 2\,x+\frac{15\,x}{5}=15\times\frac{2}{3}+15\times\frac{3}{5}$.

The next step, which is to show the cancelation, may be omitted, but it is given here to show more fully what is done.

$$15\times 2\,x+\frac{\overset{3}{\cancel{15}}\,x}{\cancel{5}}=\overset{5}{\cancel{15}}\times\frac{2}{\cancel{3}}+\overset{3}{\cancel{15}}\times\frac{3}{\cancel{5}}$$

or,

$$15\times 2\,x+3\,x=5\times 2+3\times 3$$

Next perform the multiplications.

$$30\,x+3\,x=10+9$$

$$33\,x=19$$

$$x=\tfrac{19}{33}. \quad \text{Ans.}$$

EXAMPLE 3.—Find the value of x in the equation $\frac{x+1}{3}+\frac{2\,x}{4}=47-2\,x$.

SOLUTION.—The least common denominator is 12, and when all the terms are multiplied by 12, the equation is

$$\frac{12\,(x+1)}{3}+\frac{12\times 2\,x}{4}=12\times 47-12\times 2\,x$$

from which, by cancelation,

$$4\,(x+1)+3\times 2\,x=564-24\,x$$

When the parenthesis is removed and the necessary multiplications are performed, $4\,x+4+6\,x=564-24\,x$.

The terms $-24\,x$ and 4 are now transposed and their signs changed: thus:

$$4\,x+6\,x+24\,x=564-4$$

When like terms are combined,

$$34\,x=560$$

from which

$$x=\frac{560}{34}=16\tfrac{8}{17}. \quad \text{Ans.}$$

EXAMPLE 4.—Find the value of x in the equation $\frac{32}{x+3}=\frac{18}{x}$.

SOLUTION.—The least common denominator here is the product of the two denominators, or $(x+3)x$. When each term of the equation is multiplied by this quantity the result is

$$\frac{32\,(x+3)\,x}{x+3}=\frac{18\,(x+3)\,x}{x}$$

The common factor $x+3$ can be canceled from the numerator and denominator of the first member and the common factor x can be similarly canceled from the second member. The equation will then be

$$32\,x=18\,(x+3)$$

When the terms in the parenthesis are multiplied by 18, the equation becomes

$$32\,x=18\,x+54$$

When the term $+18\,x$ is transposed to the first member it becomes $-18\,x$ and the equation then is

$$32\,x-18\,x=54$$

from which $$14\,x=54$$

and $$x=3\tfrac{12}{14}=3\tfrac{6}{7}.\quad \text{Ans.}$$

50. Order of Steps in Solving Equations.—When, in solving an equation, it is necessary to perform all the different steps just described, the order in which they are taken is as follows:

1. Clear of fractions.
2. Transpose terms.
3. Combine like terms.
4. Divide both members of the equation by the coefficient of the unknown quantity.

Sometimes the terms are transposed before the equation is cleared of fractions, as in the following example 2.

EXAMPLE 1.—Find the value of x in the equation $9\,x-9=5\,x+7$.

SOLUTION.—It was stated in Art. **18** that if a term is without a sign a plus sign is understood; thus, $5\,x$ and $9\,x$ represent $+5\,x$ and $+9\,x$.

When the unknown terms are transposed to the first member, the known terms to the second member, and the signs of the transposed terms are changed, then

$$9\,x-5\,x=7+9$$

from which $$4\,x=16$$

and $$x=4.\quad \text{Ans.}$$

EXAMPLE 2.—Find the value of the unknown quantity y in the equation $6+\frac{1}{2}\,y=8+\frac{1}{3}\,y$.

SOLUTION.—The unknown term $\frac{1}{3}y$ is first transposed to the first member, and the known term 6 to the second member, both with signs changed, and the equation is then

$$\tfrac{1}{2}y-\tfrac{1}{3}y=8-6=2$$

The members are now multiplied by the least common denominator, which is 6. Thus,

$$6\times\tfrac{1}{2}y-6\times\tfrac{1}{3}y=6\times 2$$

By cancelation, $3y-2y=12$

from which $y=12$. Ans.

EXAMPLE 3.—An iron pipe 42 feet long is cut into two parts, one being 5 times as long as the other. How long is each part?

SOLUTION.—If the shorter part is indicated by x, the other part must be $5x$, as it is five times as long as the short part. The sum of these parts, or $5x+x$, must be equal to the total length of the pipe, or 42 feet. Hence, the equation is written as follows:

$$5x+x=42$$

from which $6x=42$

and $x=\frac{42}{6}=7$ ft.

The shorter part is therefore 7 ft., and the longer part $5\times 7=35$ ft. Ans.

EXAMPLE 4.—Find the value of x in the following equation:

$$\frac{46}{x+5}=\frac{23}{x}$$

SOLUTION.—To clear of fractions the members are multiplied by the common denominator $(x+5)\,x$; thus,

$$\frac{46\,(x+5)\,x}{x+5}=\frac{23\,(x+5)x}{x}$$

The factor $x+5$ is now canceled from the fraction of the first member and x from the fraction of the second member, giving

$$46x=23\,(x+5)=23x+115$$

The term $+23x$ is now transposed; thus,

$$46x-23x=115$$

from which $23x=115$

and $x=\frac{115}{23}=5$. Ans.

EXAMPLES FOR PRACTICE

1. Find the value of the unknown quantity x in the following equations:

(*a*) $11x+14=56-3x$.
(*b*) $27x-127=11-19x$.
(*c*) $3x+4x+5x=6x+72$.
(*d*) $x-\frac{x}{3}=\frac{3}{4}+\frac{1}{2}$.
(*e*) $\frac{54}{x+4}=\frac{42}{x}$.

Ans. $\begin{cases} (a)\ x=3 \\ (b)\ x=3 \\ (c)\ x=12 \\ (d)\ x=1\frac{7}{8} \\ (e)\ x=14 \end{cases}$

2. If $x+3a=81$, and $a=17$, what is the value of x? Ans. $x=30$

3. Find the value of x in the equation $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=47$. Ans. $x=60$

4. When $3x+2a=2a+24$, what is the value of x? Ans. $x=8$

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51. Stating a Problem in the Form of an Equation.—The method to be adopted when a problem is to be stated in the form of an equation is shown by the following examples:

EXAMPLE 1.—What number must be added to 9 to give a sum of 21?

SOLUTION.—Let x represent the unknown number; the equation may be stated as follows:

$$9+x=21$$

Here, $9+x$ is the first member and 21 the second one. The equation may be solved by transposing the terms so that x stands alone on one side of the equality sign. When 9 is transposed to the second member, its sign is changed; the result will be

$$x=21-9$$

or, $$x=12. \quad \text{Ans.}$$

Because of the simplicity of the equation, it may also be solved by inspection. In this equation the value of x must be equal to the difference between 21 and 9, or 12, and if x is replaced by 12, the equation is solved, as $9+12=21$. It follows that $x=12$.

EXAMPLE 2.—What number must be added to the sum of 24 and 9 to give a sum equal to that of 38 and 5?

SOLUTION.—If the first sum and x are placed in the first member and the second sum in the second member, the equation will be as follows:

$$24+9+x=38+5$$

In this example the number of terms is too great to find the value of x conveniently by inspection. It is therefore preferable to transpose all the terms, except x, to the second member. It is necessary to change the signs of 24 and 9; the equation will then have the following form:

$$x=38+5-24-9$$

When terms with like signs are combined, $x=43-33$; and when 33 is subtracted from 43, the difference is 10; hence,

$$x=10. \quad \text{Ans.}$$

That the solution is correct may be proved by substituting 10 for x in the original equation; thus,

$$24+9+10=38+5$$

or,

$$43=43$$

Example 3.—A number when decreased by 9 is equal to 3. What is the number?

Solution.—If x is the unknown number, the problem means that if 9 is subtracted from x, the difference is 3. The equation is, therefore,

$$x-9=3$$

or,

$$x=3+9$$

from which

$$x=12. \quad \text{Ans.}$$

Example 4.—If 225 tons of coal were shipped on Tuesday, and the quantity was three times as much as that shipped on Monday, how much was shipped on the latter day?

Solution.—The unknown quantity x must be multiplied by 3 to be equal to the known amount, 225 tons. Hence,

$$3\,x=225$$

By the rule in Art. **21**, both members are divided by 3, and

$$\frac{3\,x}{3}=\frac{225}{3}$$

When 3 is canceled in the first member the equation is

$$x=\frac{225}{3}$$

or,

$$x=75 \text{ tons.} \quad \text{Ans.}$$

Example 5.—If 8 is added to the product of 9 and an unknown number, the sum is 71. Find the unknown number.

Solution.—If 8 is added to the product of 9 and the unknown number, or to $9\,x$, the sum is 71. Hence, the equation is as follows:

$$9\,x+8=71$$

The number 8 is transposed to the second member; thus,

$$9x=71-8$$

In order that x may be alone in the first member, both members are divided by 9. Hence,

$$\frac{9x}{9}=\frac{71-8}{9}$$

or,

$$x=\frac{71-8}{9}=\frac{63}{9}=7. \quad \text{Ans.}$$

APPLICATION OF FORMULAS

SOLUTIONS OF SIMPLE STANDARD FORMULAS

SUBSTITUTION IN FORMULAS

52. Application of Standard Forms of Formulas.—The formulas used in mathematical work are equations that express, briefly and concisely, rules or methods that would be difficult to understand and apply if expressed in words. When a formula is expressed so that the letter representing the unknown quantity in the problem is the only letter in the first member of the equation, the application of the formula is simple. It is then necessary only to substitute the given values for the other letters and to perform the indicated operations.

In several of the preceding articles simple substitutions which illustrate the method have been made. These show that the given values are written in place of the letters. Care must be taken to indicate the correct operations between the numbers. It must be remembered that when two letters are written together with no sign between them multiplication is really indicated. For example, ab means $a\times b$ and if $a=5$ and $b=4$, $ab=5\times4$ or 20. If a letter has a coefficient, a multiplication sign must be written between the coefficient and the number which is substituted for the letter. Thus, if $a=3$, $4a=4\times3$ since $4a$ means $4\times a$. Similarly, $6a^2=6\times3^2$ or $6\times3\times3$.

The following rule is applied in mensuration: "The volume of a pyramid or a cone equals the area of the base multiplied by one-third of the altitude." This rule may be expressed by a formula, as follows:

$$V=\frac{a\,b}{3}$$

in which V = volume, in cubic inches;
a = altitude, in inches;
b = area of base, in square inches.

Or, if a is taken as the altitude in *feet* and b as the area of the base in *square feet*, then V will be the volume in *cubic feet*.

Suppose the area of the base of a pyramid is 8 square inches and the altitude is 6 inches. The volume is found by substituting 8 for b and 6 for a. Then,

$$V=\frac{6\times 8}{3}=16$$

That is, the required volume is 16 cubic inches.

EXAMPLES FOR PRACTICE

1. Find the value of A in the formula $A=b\,h$, for the following cases

(*a*) $b=9$, $h=7$.
(*b*) $b=17$, $h=9$.

Ans. (*a*) 63; (*b*) 153

2. Given that $A=\frac{1}{2}\,h\,b+\frac{1}{2}\,h\,b'$. Find the value of A when

(*a*) $b=5$, $b'=7$, and $h=4$.
(*b*) $b=6$, $b'=8$, and $h=5$.

Ans. (*a*) 24; (*b*) 35

3. If $a=1$, $b=3$, and $c=5$, find the value of the following:

(*a*) $a^2+2\,b^2+3\,c^2$.
(*b*) $a^2+2\,a\,b+b^2$.

Ans. (*a*) 94; (*b*) 16

TRANSFORMATION OF FORMULAS

53. Formulas Requiring Transformation.—The formula stated in the preceding article gives the volume of a pyramid or a cone, but the volume and the altitude may be known and it may be necessary to find the area of the base. For instance, it might be necessary to find the area of the base of a cone which is 18 inches high and which must contain 10 cubic inches. Then V and a are the known quantities and b is to be determined. The formula

$$V=\frac{a\,b}{3} \qquad (1)$$

must then be transformed so that the symbol b is alone on one side of the equality sign, preferably at the left. To make this transformation, or to solve the equation to find the value of b, the order of steps given in Art. **50** is followed. First clear the equation of fractions by multiplying both members by 3. Then,

$$3\,V=a\,b$$

Next divide both members by a, the coefficient of b. The formula then becomes

$$\frac{3\,V}{a}=b$$

or,

$$b=\frac{3\,V}{a} \qquad (2)$$

From this formula, the area of the base may be found, if the volume and the altitude are known.

Should it be necessary to calculate the altitude of a pyramid with a given volume and base area, formula **1** must be transformed so that the symbol a constitutes the first member of the formula.

The various steps to be taken are the same as in the preceding case. Thus, both members are multiplied by 3; then

$$3\,V=a\,b$$

Both members are next divided by b, and the formula is

$$\frac{3\,V}{b}=a$$

or,

$$a=\frac{3\,V}{b} \qquad (3)$$

Formula **1** is thus used as a basis for developing two new formulas to serve other conditions.

EXAMPLE.—The volume of a pyramid must be equal to 220 cubic inches. If the altitude is 18 inches, what must be the area of the base?

SOLUTION.—By formula 2, with the given values, $V=220$ and $a=18$, substituted, the formula is:

$$b=\frac{3\times 220}{18}$$

from which $b=\frac{220}{6}=36.67$ sq. in., nearly. Ans.

54. Scope of Examples.—A series of examples will now be given showing how transformations are made in some of the formulas that may be encountered in succeeding Sections. It is to be borne in mind, however, that these formulas are chosen merely as examples to show how they and similar formulas may be transformed; no knowledge of the practical application is required in connection with the examples.

These formulas are to be studied simply as illustrations of the way in which the principles of solving equations are applied in transforming formulas. The use of any particular formula will be explained in detail in the Course with which it is connected; therefore, no attempt need be made at this time to understand its application or the technical terms connected with it.

FORMULAS RELATING TO ELECTRICAL AND STEAM ENGINEERING

55. Ohm's Law.—One of the fundamental formulas of electrical engineering is known as **Ohm's law,** which states the relations between the strength of an electric current, its pressure, or electromotive force, and the resistance of the electric circuit.

Let I = current strength, in amperes;
E = electromotive force, or pressure, in volts;
R = resistance, in ohms.

Then, $$I=\frac{E}{R} \qquad (1)$$

If both members of the equation are multiplied by R, then

$$IR=\frac{ER}{R}$$

By cancelation, $IR=E$

or, $E=IR$ (2)

If the members of formula **2** are divided by I,

$$\frac{E}{I}=\frac{IR}{I}$$

By cancelation, $\frac{E}{I}=R$

or, $R=\frac{E}{I}$ (3)

Example 1.—The electromotive force of a current is 3 volts. If the resistance of the electric circuit is 15 ohms, what is the current strength, in amperes?

Solution.—By formula **1**, $I=\frac{E}{R}$ and, with the known values substituted, $I=\frac{3}{15}=.2$ ampere. Ans.

Example 2.—What is the resistance of an electric circuit in which a pressure of 110 volts produces a current of 10 amperes?

Solution.—By formula **3**, $R=\frac{E}{I}=\frac{110}{10}=11$ ohms. Ans.

Example 3.—In an electric circuit of 8 ohms resistance flows a current of 6 amperes. What is the pressure, or electromotive force?

Solution.—By formula **2**, $E=IR=6\times8=48$ volts. Ans.

56. Steam-Engine Formula.—The rule for finding the horsepower of a steam engine is generally given in the following form:

Rule.—*The horsepower of an engine equals the product of the average steam pressure on the piston, in pounds per square inch, the length of the stroke, in feet, the area of the piston, in square inches, and the number of strokes per minute, divided by 33,000.*

This is a very simple rule and easy to remember, yet it is more common to express it as a formula, which is usually written thus:

$$H=\frac{PLAN}{33{,}000} \qquad (1)$$

where H = horsepower of engine;
P = average steam pressure per square inch of piston;
L = length of stroke of piston, in feet;
A = area of piston, in square inches;
N = number of working strokes of piston, per minute.

The number 33,000 is known as a *constant*, because it is a fixed value used in power calculations. Various constants are employed in formulas.

In order that the horsepower of an engine may be found, it is necessary to know the average steam pressure per square inch of the piston; the area of the face of the piston, in square inches; the length of the stroke of the piston, in feet; and the number of working strokes of the piston, per minute. In problems relating to the steam engine, the stroke is often given in inches and then it must be reduced to feet, by dividing by 12, when used in the formula. The number of revolutions per minute is usually given instead of the number of strokes, in which case the number of revolutions must be multiplied by the number of working strokes per revolution, which depends on the design of the engine. Unless otherwise stated, an engine is assumed to be a double stroke engine; that is, the number of working strokes per revolution is 2. It is also customary to give the diameter of the cylinder instead of its area. As the piston is circular, its area must be found from the rule given in a preceding Section, namely, *the area of a circle is equal to .7854 times the square of the diameter*, which is more easily remembered as the formula,

$$A = .7854\ D^2 \qquad (2)$$

in which A = area of circle, in square inches;
D = diameter, in inches.

Example.—What is the horsepower of an engine, if the piston is 20 inches in diameter, the stroke 36 inches long, the average steam pressure 40 pounds per square inch, and the engine makes 80 revolutions per minute?

Solution.—In order to apply the formula for horsepower, it is first necessary to find the area of the piston. By formula **2**, $A = .7854\ D^2 = .7854 \times 20 \times 20 = 314.16$ sq. in. L, the length of the stroke, must be reduced to feet; hence, $L = 36 \div 12 = 3$ ft. The number of strokes per minute N is $80\times2=160$. $P=40$ lb. per sq. in. Hence, when these known values are substituted in formula **1**,

$$H=\frac{40\times3\times314.16\times160}{33{,}000}=182.78 \text{ H. P.} \quad \text{Ans.}$$

57. Transformation of Steam-Engine Formula.—Sometimes the horsepower of the engine is given, and likewise the average steam pressure, the length of the stroke, and the number of revolutions per minute, and it is required to find the diameter of the steam cylinder. In that case, formula **1**, Art. **56**, must be transformed so that the area A, which governs the diameter, will be in the first member, or on the left-hand side of the formula, and the other terms on the right-hand side.

Both sides of the formula are multiplied by 33,000 and

$$33{,}000\, H=P\,L\,A\,N$$

If the second member is divided by $P\,L\,N$, it becomes $\frac{P\,L\,A\,N}{P\,L\,N}$ or A. Therefore, both members are divided by $P\,L\,N$ to make A the only letter in the second member. The formula then becomes

$$\frac{33{,}000\, H}{P\,L\,N}=A$$

or,

$$A=\frac{33{,}000\, H}{P\,L\,N} \qquad (\mathbf{1})$$

The area can now be found from formula **1**, and the diameter by using formula **2**, Art. **56**. It is more convenient, however, to substitute $.7854\, D^2$ for A in the formula just found; thus,

$$.7854\, D^2=\frac{33{,}000\, H}{P\,L\,N}$$

Since D is to be found, divide both members by .7854; the coefficient in the second member will be $33{,}000\div.7854$ or 42,017 and the literal part will be $\frac{H}{P\,L\,N}$. Therefore,

$$D^2=\frac{42{,}017\, H}{P\,L\,N}$$

The square root of both members is now extracted.

$$D=\sqrt{\frac{42{,}017\,H}{P\,L\,N}}$$

The square root of 42,017 is 205, nearly; hence,

$$D=205\sqrt{\frac{H}{P\,L\,N}} \qquad (2)$$

This is read *D equals 205 times the square root of H over P L N.*

EXAMPLE.—What should be the diameter of the cylinder of a 100-horsepower steam engine to work with a mean effective pressure of 42 pounds per square inch, a stroke of 30 inches, and make 100 revolutions per minute?

SOLUTION.—Here $H=100$, $P=42$, $L=\frac{30}{12}=2.5$, and $N=2\times100=200$. When these values are substituted in formula **2**, it becomes

$$D=205\sqrt{\frac{H}{P\,L\,N}}=205\sqrt{\frac{100}{42\times2.5\times200}}$$

The first step is to simplify the fraction under the radical sign.

$$D=205\sqrt{\frac{100}{42\times2.5\times200}}=205\sqrt{.004762}$$

$$=205\times.069=14.145 \text{ in., nearly.} \quad \text{Ans.}$$

In practice, the nearest common fraction of an inch would be chosen instead of .145, as $\frac{1}{8}$, and the answer would be $14\frac{1}{8}$ in., nearly.

58. Suppose it is required to transform formula **1**, Art. **56**, so that the value of *N* may be found, if the other quantities are known.

First multiply both members of the formula $H=\frac{P\,L\,A\,N}{33{,}000}$ by 33,000; thus,

$$33{,}000\,H=P\,L\,A\,N$$

To obtain *N* alone on one side of the equality sign, divide both members by *P L A*; thus,

$$\frac{33{,}000\,H}{P\,L\,A}=\frac{P\,L\,A\,N}{P\,L\,A}$$

By cancelation,

$$\frac{33{,}000\,H}{P\,L\,A}=N$$

or,

$$N=\frac{33{,}000\,H}{P\,L\,A} \qquad (1)$$

Example.—The piston area of a steam engine is 78.54 square inches, the mean effective pressure is 40 pounds per square inch of piston, and the length of stroke is $1\frac{1}{4}$ feet. If the engine is to develop 60 horsepower, what must be the number of strokes per minute?

Solution.—Here $H=60$, $P=40$, $L=1\frac{1}{4}$, and $A=78.54$, and when these known values are substituted in formula **1**, it becomes

$$N=\frac{33{,}000\times 60}{40\times 1\frac{1}{4}\times 78.54}=504.2 \text{ strokes per minute. Ans.}$$

FORMULAS RELATING TO MENSURATION AND MINE VENTILATION

59. Area of a Triangle.—A good example of a formula using powers and roots of numbers is shown in the following formula for finding the area of a triangle when the three sides are known:

$$A=\frac{b}{2}\sqrt{a^2-\left(\frac{a^2+b^2-c^2}{2b}\right)^2}$$

in which A = area of triangle; a, b, and c = lengths of sides. If a, b, and c are in inches, A will be in square inches; and if a, b, and c are in feet, A will be in square feet.

A simpler formula which will give the area of a triangle is $A=\sqrt{s(s-a)(s-b)(s-c)}$, in which A, a, b, and c have the same meaning as in the preceding example and $s=\frac{a+b+c}{2}$. To illustrate the use of powers the following example will be solved by the more complicated formula.

Example.—What is the area of a triangle whose sides are 21 feet, 46 feet, and 50 feet long?

Solution.—It makes no difference which side is represented by a, b, or c; hence, let a represent the side 50 ft. long; b, the side 46 ft. long; and c the side 21 ft. long. Then, by the formula,

$$A=\frac{b}{2}\sqrt{a^2-\left(\frac{a^2+b^2-c^2}{2b}\right)^2}=\frac{46}{2}\sqrt{50^2-\left(\frac{50^2+46^2-21^2}{2\times 46}\right)^2}$$

$$=\frac{46}{2}\sqrt{2{,}500-\left(\frac{2{,}500+2{,}116-441}{92}\right)^2}=23\sqrt{2{,}500-\left(\frac{4{,}175}{92}\right)^2}$$

$$=23\sqrt{2{,}500-(45.38)^2}=23\sqrt{2{,}500-2{,}059.34}=23\sqrt{440.66}$$

$$=23\times 20.992=482.8 \text{ sq. ft., nearly. Ans.}$$

The operations in this example have been extended much further than is necessary, in order to show every step of the process. It will be found instructive to change the numbers of the different sides so that a represents either the 21- or the 46-foot side and b and c the other sides. The values so taken can then be substituted in the formula and the operations performed as before; the answer will be the same as that already found with a slight variation depending on the number of decimal places to which the results are carried.

60. Area of a Segment of a Circle.—A formula for finding the area A of a segment of a circle is $A=\frac{4h^2}{3}\sqrt{\frac{D}{h}-.608}$, where h is the height of the segment and D is the diameter of the circle.

EXAMPLE.—Find the area of a circular segment if the height h equals 18 inches and the diameter D of the circle equals 60 inches.

SOLUTION.—Here $h=18$ and $D=60$. When these values are substituted in the formula, it becomes

$$A=\frac{4\times 18^2}{3}\sqrt{\frac{60}{18}-.608}$$

First find the numerical value of $\frac{60}{18}-.608$ and then extract its square root.

$$\frac{60}{18}-.608=3.333-.608=2.725$$

$$\sqrt{2.725}=1.65$$

Next find the value of $\frac{4\times 18^2}{3}$. This equals $\frac{4\times 324}{3}=432$.

Then, $A=432\times 1.65=712.8$ sq. in. Ans.

61. Mine-Ventilation Formulas.—In calculations relating to mine ventilation it is necessary to ascertain the quantity of air flowing per minute through an airway under certain conditions. The rule to be applied is as follows:

Rule.—*The product of the sectional area of airway, in square feet, and the velocity of air-current, in feet per minute, equals the quantity of air in circulation, in cubic feet per minute.*

The rule written as a formula is,

$$q = a\,v \qquad (1)$$

in which q = quantity of air in circulation, in cubic feet per minute;
a = sectional area of airway, in square feet;
v = velocity of air-current, in feet per minute.

EXAMPLE.—What quantity of air will be delivered per minute through a 6′×8′ airway, if the air travels at the rate of 500 feet per minute?

SOLUTION.—By the preceding formula, in which the following known values are substituted, namely, $a = 6 \times 8 = 48$ sq. ft. and $v = 500$, it is found that $q = a\,v = 48 \times 500 = 24{,}000$ cu. ft. per min. Ans.

62. If it is required to find the velocity of the air-current, when the sectional area of the airway and the circulating volume of air is known, formula **1**, Art. **61**, must be transformed in the following manner:

Divide both members of the formula $q = a\,v$ by a; thus,

$$\frac{q}{a} = \frac{a\,v}{a}$$

Cancel a in the second member, and

$$\frac{q}{a} = v$$

or,

$$v = \frac{q}{a}$$

EXAMPLE.—If the quantity of circulating air is 24,000 cubic feet per minute and the airway is 6 ft.×8 ft., what is the velocity of the air-current, in feet per minute?

SOLUTION.—By substituting in the formula a value for $q = 24{,}000$ and for $a = 6 \times 8 = 48$,

$$v = \frac{24{,}000}{48} = 500 \text{ ft. per min.} \quad \text{Ans.}$$

63. The **rubbing surface** of an airway is the entire inner surface of the airway composed of the roof, floor, and sides. It is estimated in square feet. To find the area of this rubbing surface the following rule is used:

Rule.—*The area of the rubbing surface of the entire airway equals the perimeter, that is, the distance around the cross-section of the airway, multiplied by the length of the airway.*

The formula for this rule is

$$s = l\,o$$

in which s = area of rubbing surface, in square feet;
l = length of airway, in feet;
o = perimeter, in feet.

Example.—What is the area of the surface of an airway 6 ft.×8 ft. and 1,250 feet long?

Solution.—Here $o=6+6+8+8=28$ ft.; that is, the distance around the section of the airway equals the sum of the lengths of its four sides. The length $l=1{,}250$ ft. When these values are substituted for o and l in the formula, $s=1{,}250\times 28=35{,}000$ sq. ft. Ans.

64. The pressure necessary to produce a certain velocity of air-current in an airway of given dimensions is as follows:

Rule.—*The pressure, in pounds per square foot, required to produce a given velocity of the air-current, equals the product of the coefficient of friction, the area of the rubbing surface of the airway, and the square of the velocity of the flow, divided by the sectional area of the airway.*

The rule, given as a formula, is

$$p = \frac{k\,s\,v^2}{a}$$

in which p = ventilating pressure, in pounds per square foot;
k = coefficient of friction, which for mine passages may be taken as .00000002;
s = rubbing surface, in square feet;
v = velocity of air-current, in feet per minute;
a = sectional area of airway, in square feet.

It should be noted that s, v, and a of this formula represent the same quantities as similar letters in Arts. **61** and **63**.

Example.—What pressure is required to produce a circulation with a velocity of 500 feet per minute in an airway 6 ft.×8 ft. and 1,250 feet long?

Solution.—Here $k=.00000002$, $s=lo=1{,}250\times 28=35{,}000$ sq. ft., $v^2=500\times 500=250{,}000$, and $a=6\times 8=48$ sq. ft. Hence, by the formula,

$$p=\frac{.00000002\times 35{,}000\times 250{,}000}{48}=3.646 \text{ lb. per sq. ft.}\quad \text{Ans.}$$

65. At times no one formula is in a suitable form for finding a required quantity; that is, the known quantities are not entirely the same as those required by any formula. In such cases it may be necessary to combine two or more formulas to get one in a form suitable for employing the known quantities.

For example, the formula $p=\frac{k s v^2}{a}$ requires the quantities a, s, and v to be known. If, now, instead of s and v, the quantities l, o, and q are known, it is possible to employ the formulas $v=\frac{q}{a}$ and $s=l o$, Arts. **62** and **63**, and by this means obtain the data required by the formula $p=\frac{k s v^2}{a}$. Thus, instead of using s and v in this formula, the equivalent values may be employed in their places. Hence, instead of s, the term $l o$ is inserted and instead of v, the term $\frac{q}{a}$; but as v is squared, the term $\frac{q}{a}$ must likewise be squared and written $\frac{q^2}{a^2}$.

The substitution of these terms is done in the following manner:

$$p=\frac{k s v^2}{a}=\frac{k\times l o\times\frac{q^2}{a^2}}{a}=k\times l o\times\frac{q^2}{a^2}\times\frac{1}{a}=\frac{k l o q^2}{a^3}$$

Example.—What pressure is required to produce a circulation of 24,000 cubic feet of air per minute in an airway 6 ft.×8 ft. in cross-section and 1,250 feet long?

Solution.—In this problem, $q=24{,}000$, $l=1{,}250$, $o=2(6+8)=28$, $a=6\times8=48$, $k=.00000002$. Then, by the formula just found,

$$p=\frac{.00000002\times1{,}250\times28\times24{,}000\times24{,}000}{48\times48\times48}=3.646 \text{ lb. per sq. ft.}\quad \text{Ans.}$$

FORMULAS RELATING TO NAVIGATION

66. Fuel Consumption and Speed.—The fuel consumption may be said to vary directly as the horsepower developed (this is not exactly true, but only approximately). The horsepower varies directly as the cube of the speed, whence it follows that the fuel consumption also will vary as the cube of the speed (approximately).

Let S = certain speed of vessel;
C = coal consumption at speed S;
s = new speed;
c = coal consumption at speed s.

Then, the following formula may be derived:

$$c = \frac{s^3 C}{S^3}$$

EXAMPLE.—A steamer consumes 80 tons of coal per day at a speed of 12 knots. Suppose that the speed is to be reduced to 10 knots. What will be the fuel consumption per day at that rate of speed?

SOLUTION.—In this problem $s = 10$, $C = 80$, and $S = 12$. By substituting these values for s, C and S in the formula it is found that

$$c = \frac{10^3 \times 80}{12^3} = \frac{1{,}250}{27} = 46.3 \text{ T. per da.} \quad \text{Ans.}$$

FORMULAS RELATING TO BEAMS AND COLUMNS

67. Formula for Wooden Beam.—The following formula is used when it is desired to obtain the size of a rectangular wooden beam that will safely support a given load, uniformly distributed over the beam:

$$\frac{12\,W\,L}{8} = \frac{s}{f} \times \frac{b\,d^2}{6}$$

in which W = total uniform load on the beam, in pounds;
L = length, or span, of beam, in feet;
s = unit fiber stress of the material of which the beam is composed;
f = factor of safety;
b = breadth of beam, in inches;
d = depth of beam, in inches.

EXAMPLE.—What must be the depth of a rectangular beam of yellow pine to carry a total safe uniformly distributed load of 8,000 pounds, if the span is 15 feet and the width of the beam is 6 inches? It is assumed that the fiber stress s is 7,000 pounds per square inch, and that the factor of safety f is 6.

SOLUTION.—The formula $\frac{12WL}{8}=\frac{s}{f}\times\frac{bd^2}{6}$ may be written

$$\frac{12WL}{8}=\frac{bd^2s}{6f}$$

or,

$$\frac{bd^2s}{6f}=\frac{12WL}{8}$$

As d is the unknown quantity, the formula should be transformed so that d is alone on one side of the equality sign.

To clear of fractions, multiply both members by $24f$.

$$4bd^2s=36WLf$$

Divide both members by $4bs$ to obtain d^2 alone.

$$d^2=\frac{9WLf}{bs}$$

and

$$d=\sqrt{\frac{9WLf}{bs}}$$

from which

$$d=\sqrt{\frac{9\times 8{,}000\times 15\times 6}{6\times 7{,}000}}=\sqrt{154.29}=12.4\text{ in.}$$

As 12.4 in. is not a stock size of timber, the next larger stock size, which is 14 in., should be chosen.

68. Reinforced-Concrete Columns.—The following formula may be used for designing reinforced-concrete columns:

$$P=S_c(A+na)$$

in which P = total safe load on the column, in pounds;

S_c = safe unit stress, in pounds, produced in the concrete by load P;

A = area of concrete in column section, in square inches;

n = ratio of modulus of elasticity of steel to modulus of elasticity of concrete, which is about 15;

a = area of steel in column section, in square inches.

EXAMPLE.—Design a reinforced-concrete column to withstand safely a load of 100,000 pounds. If $S_c=400$ pounds and $n=15$, what must be the sectional area of the concrete and steel, if the sectional area of the steel is 3 per cent. of that of the concrete? That is,

$$a=\frac{3}{100}A$$

SOLUTION.—The simplest way to solve this example is to substitute in the given formula the values known. Also write $\frac{3}{100}A$ in place of a.

Then,
$$100{,}000=400\left(A+15\times\frac{3}{100}A\right)$$
or,
$$100{,}000=400\left(A+\frac{45}{100}A\right)$$

Both members can be divided by 400 to avoid large numbers in the solution. The equation then becomes

$$250=A+\frac{45}{100}A$$

To combine the coefficients of A first reduce $\frac{45}{100}$ to the decimal .45. Then, since A has the same value as $1A$, $A+.45A=1.45A$. Thus,

$$250=1.45A$$

To get the value of A, divide both members by 1.45, the coefficient of A.

$$A=\frac{250}{1.45}=172.41 \text{ sq. in.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. What is the area of a circle 17 feet in diameter? Ans. 227 sq. ft.

2. A steam engine has a single cylinder 18 inches in diameter and a 30-inch stroke. It uses steam on both sides of the piston, runs at 84 revolutions per minute, and has a mean effective pressure of 45 pounds per square inch. What is the horsepower of the engine? Ans. 145.74 H. P.

3. If 80,000 cubic feet of air per minute flows through an 8′×8′ airway, what is the velocity of flow? Ans. 1,250 ft. per min.

4. What is the surface of an 8′×8′ airway 1,600 feet long? Ans. 51,200 sq. ft.

5. A current of 20 amperes is flowing under a pressure of 220 volts. What is the resistance, in ohms, of the electric circuit? Ans. 11 ohms

6. In the formula $x=\frac{d+c^2}{d^2-40}$ substitute the following values: $d=10$, $c=2$. What is the value of x? Ans. $x=\frac{7}{30}$

7. If D, the diameter of a circle, is 54 inches, and h, the height of a segment, is 20 inches, find the area of the segment by applying the formula in Art. **60**. Ans. 771.2 sq. in.

8. The formula for finding the surface area of a sphere is $A=D^2\times 3.1416$. If D, the diameter of the sphere, is 14 inches, what is the area? Ans. 615.75 sq. in.

CUBE ROOT

Serial 1983–2 Edition 1

EXTRACTION OF CUBE ROOT

METHODS AVAILABLE

INTRODUCTION

1. Definition.—The **cube root** of a number is one of the three equal factors of that number. For example, 4 is the cube root of 64, because $4\times4\times4=64$; that is, 4 is one of three equal factors that, when multiplied together, produce 64. Similarly, 3 is the cube root of 27, because $3\times3\times3=27$. Again, 2 is the cube root of 8, because $2\times2\times2=8$; 6 is the cube root of 216, because $6\times6\times6=216$; and so on.

2. Indicating Cube Root.—The cube root of any number is indicated by writing the radical sign ($\sqrt{\ }$) in front of it, the vinculum ($\overline{\quad}$) over it, and the index figure ^{3} above and to the left of the radical sign. For example, $\sqrt[3]{125}$ indicates the cube root of 125; $\sqrt[3]{15.625}$ indicates the cube root of 15.625; $\sqrt[3]{\frac{27}{343}}$ indicates the cube root of $\frac{27}{343}$; and so on.

3. Extracting Cube Root.—The operation of finding the cube root of a number is called the *extraction* of the cube root; that is, the cube root of a number is extracted when one of the three equal factors of that number is found. The extraction of the cube roots of numbers must frequently be done in the solution of certain kinds of problems that are met with in engineering work. For this reason it is necessary that

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instruction be given as to how to find the cube roots of numbers.

4. Methods of Finding Cube Root.—There are a number of different ways in which the cube root of a number may be found, but in this Section only three methods will be described. These will be called (*a*) the trial-and-error method, (*b*) the table method, and (*c*) the exact method, and they will be described in detail in later articles.

5. Significant Figures.—In any number, the figures beginning with the first digit at the left and ending with the last digit at the right, are called the **significant figures** of the number. Thus, the number 405,800 has the four significant figures 4, 0, 5, 8; and the number .000090067 has the five significant figures 9, 0, 0, 6, and 7. A cipher is not a digit. Consequently, if a number begins or ends with ciphers, they are not considered in determining the significant figures of that number. But if ciphers occur between digits, as in the numbers 2,807 and 21.03, they are then considered as significant figures.

The part of a number consisting of its significant figures is called the **significant part** of the number. Thus, in the number 28,070, the significant part is 2807; in the number .00812, the significant part is 812; and in the number 170.3, the significant part is 1703.

The significant figures or the significant part of a number consists of the figures, in their proper order, from the first digit at the left to the last digit at the right, without regard to the position of the decimal point. Hence, *all numbers that differ only in the position of the decimal point have the same significant part.* For example, .002103, 210.3, 21,030, and 210,300 have the same significant figures 2, 1, 0, and 3, and the same significant part 2103.

6. Exact and Approximate Cube Roots.—Only a comparatively few numbers can be separated into exactly equal factors. For example, considering all the numbers from 1 to 1,000, there are only ten of which the exact cube root can be found, namely, 1, 8, 27, 64, 125, 216, 343, 512, 729, and 1,000.

All the other numbers between 1 and 1,000 have only approximate cube roots; that is, their cube roots cannot be found exactly, because those roots will contain unending decimals. For example, suppose that the cube root of 50 is to be found. There are no three equal numbers that, if multiplied together, will produce 50; but 3.6 is an approximate cube root of 50, because 3.6×3.6×3.6=46.656. If the root is carried to another decimal place, or made 3.68, the result is even better, because 3.68×3.68×3.68=49.836032, which is very close to 50. If another decimal place is added and the root is made 3.684, the

TABLE I

PERFECT CUBES AND EXACT CUBE ROOTS

Number	Cube Root	Number	Cube Root	Number	Cube Root	Number	Cube Root
1	1	17,576	26	132,651	51	438,976	76
8	2	19,683	27	140,608	52	456,533	77
27	3	21,952	28	148,877	53	474,552	78
64	4	24,389	29	157,464	54	493,039	79
125	5	27,000	30	166,375	55	512,000	80
216	6	29,791	31	175,616	56	531,441	81
343	7	32,768	32	185,193	57	551,368	82
512	8	35,937	33	195,112	58	571,787	83
729	9	39,304	34	205,379	59	592,704	84
1,000	10	42,875	35	216,000	60	614,125	85
1,331	11	46,656	36	226,981	61	636,056	86
1,728	12	50,653	37	238,328	62	658,503	87
2,197	13	54,872	38	250,047	63	681,472	88
2,744	14	59,319	39	262,144	64	704,969	89
3,375	15	64,000	40	274,625	65	729,000	90
4,096	16	68,921	41	287,496	66	753,571	91
4,913	17	74,088	42	300,763	67	778,688	92
5,832	18	79,507	43	314,432	68	804,357	93
6,859	19	85,184	44	328,509	69	830,584	94
8,000	20	91,125	45	343,000	70	857,375	95
9,261	21	97,336	46	357,911	71	884,736	96
10,648	22	103,823	47	373,248	72	912,673	97
12,167	23	110,592	48	389,017	73	941,192	98
13,824	24	117,649	49	405,224	74	970,299	99
15,625	25	125,000	50	421,875	75	1,000,000	100

accuracy is still further increased, because $3.684 \times 3.684 \times 3.684 = 49.998717504$. But no matter to how many decimal places the root may be extended, the cube of that root will never be exactly 50, although it will approach nearer and nearer to 50. If an exact root cannot be found, therefore, the approximate root should be carried to as many significant figures as the accuracy of the solution demands.

7. Table I shows all the numbers between 1 and 1,000,000, inclusive, that have exact cube roots, these roots being the whole numbers from 1 to 100, inclusive. The larger numbers are the cubes of the corresponding smaller numbers, and any number that is a perfect cube has an exact cube root.

8. Preliminary Steps in Finding Cube Root.– If the cube root of a number is to be found, the first step is to point off the number into periods of three figures, beginning with the units place if the number is a whole number, and at the decimal point if the number is a decimal. If it is a mixed number, the pointing off is begun at the decimal point and is carried to the right and to the left. These instructions may be made clear by illustrative examples. Suppose, for example, that the cube root of 405,224 is to be found. The number is pointed off into periods, or parts, containing three figures, beginning at the right, or at the units place. When thus pointed off, which may be indicated by prime marks ′ or by commas, the number becomes 405′224. The reason for pointing off in this way is to determine how many figures there are in the root. For every period, or part, there will be one figure in the root; and as 405′224 contains two such periods, the cube root consists of two figures. Table I shows that the cube root of 405,224 is 74, which consists of two figures.

9. It may happen that, when the number whose cube root is to be found is pointed off, the first period at the left will contain one or two figures instead of three. In such a case, the one or two figures are considered as a full period. For example, suppose that the number whose cube root is to be found is 91,125. When pointed off it becomes 91′125, and as the first two figures constitute a period, there are two periods in the number, which indicates that the cube root contains two figures. Table I proves the correctness of this conclusion, as the cube root of 91,125 is 45.

If the number 5,832 is pointed off, it becomes 5′832, in which the first period contains but one figure. However, as there are two periods, the cube root contains two figures. Reference to Table I shows this to be the case, as the cube root of 5,832 is 18.

10. If the number whose cube root is to be found is a decimal, it is divided into periods of three figures each, *beginning at the decimal point.* In case the last period at the right contains only one or two figures, ciphers must be annexed until the period has three figures. The cube root then contains as many figures, following the decimal point, as there are periods in the original number. Annexing ciphers to the right of a decimal, as explained in a preceding Section, does not alter the value of the decimal in any way. It is done in this connection simply to fill out the period that contains less than three figures. For example, if the cube root of .14625 were required, it would be pointed off thus, .146′25, and as the last period has only two figures, a cipher would be annexed, making .146′250. As this number contains two periods, the cube root of it will be a decimal and will contain two figures, one for each period; but as the number .14625 is not a perfect cube, its cube root will not be exact and so the root may be carried to any desired number of decimal places, according to the accuracy required. This is done by annexing extra periods of three ciphers each.

11. If the cube root of a mixed number is to be found, the number is divided into periods of three figures each, beginning at the decimal point and proceeding in each direction. If the last period at the *right* has less than three figures, ciphers are annexed as in the case of a decimal number. For example, suppose that 1,274.4285 is the number to be considered. When pointed off it becomes 1′274.428′5, and ciphers must be annexed to the last period at the right, giving 1′274.428′500. As there are four periods, the root will contain four figures, two preceding the decimal point and two following it. In the number just considered, the decimal point serves to separate the second and third periods, and no other mark of separation is needed.

TRIAL-AND-ERROR METHOD

12. Roots of Whole Numbers and Decimals.—In solving practical problems that require the finding of cube root, it is usually not necessary to obtain the root to more than four significant figures, and in most instances three significant figures will be enough. For such approximate calculations, therefore, a simple way of finding the cube root of a number is by trial and error. As explained in Art. **1**, the cube root of a number is one of the three equal factors that, when multiplied together, will produce the number. The trial-and-error method, therefore, consists simply in assuming a root and cubing it. If the product is greater than the number whose root is to be found, the assumed root is made smaller and the cubing operation is repeated. By making a number of trials in this way, eventually a root is found that, when cubed, will give a product very nearly equal to the given number. That root is then taken as the cube root of the number. The method may be illustrated most clearly by examples and their solutions.

EXAMPLE 1.—Find the cube root of 4,913 by the trial-and-error method.

SOLUTION.—The number is first pointed off into periods, as explained in Art. **8**, which gives 4′913. As there are two periods, the cube root contains two figures. The first period consists of one figure, 4. Now, the first figure of the root cannot be 2, because $2\times2\times2=8$, which is much greater than 4. The smallest root of two figures, beginning with 2, is 20, and $20^3=20\times20\times20=8{,}000$, which is considerably greater than the given number, 4,913. So it is plain that the root must be less than 20. Suppose that 18 is assumed; then, $18\times18\times18=5{,}832$, which is also somewhat larger than 4,913, showing that the root must be less than 18. Hence, 17 is chosen; then $17\times17\times17=4{,}913$, the given number. Therefore, 17 is the exact cube root of 4,913. Ans.

EXAMPLE 2.—By the trial-and-error method find the cube root of 12,764.

Solution.—When pointed off properly, the number becomes 12′764. Reference to Table I shows that the first figure of the root must be 2, because $2^3=8$, which is less than 12, the first period, whereas $3^3=27$, which is greater than the first period. If 12′764 has an exact root, there must be two figures in it, because the number has two periods, and it has just been shown that the first figure is 2. So, let 22 be chosen as a trial root; then, $22\times22\times22=10{,}648$, which is less than 12,764, showing that the root must be greater than 22. Let 23 be chosen next; then, $23\times23\times23=12{,}167$. This is also less than 12,764. So let 24 be chosen; then, $24\times24\times24=13{,}824$. This is considerably larger than the given number, 12,764, and indicates that the cube root cannot be 24. But it was also found that 23 was a little too small. Therefore, the root must be greater than 23 and less than 24. So, take 23.4 as a trial root; then, $23.4\times23.4\times23.4=12{,}812.904$. This is also larger than 12,764, showing that the root must be less than 23.4. Let 23.38 be chosen; then, $23.38\times23.38\times23.38=12{,}780.078472$, which is so close to 12,764 that it may be considered approximately equal to it. Then, 23.38 is the cube root of 12,764, very nearly. Ans.

Example 3.—What is the value of $\sqrt[3]{.0837}$ by the trial-and-error method?

Solution.—The number is first pointed off, and ciphers are annexed to complete the last period, giving .083′700. The given number is wholly decimal and so the root will be wholly decimal. As there are two periods, there will be two figures following the decimal point in the root, if the root is exact, and more if the root is not exact. The first period contains the number 83. Reference to Table I shows that the first figure of the root must be 4, because $4^3=64$, which is less than 83, whereas $5^3=125$, which is greater than 83. Hence, it is known that the root must be a number greater than .40. Let .45 be assumed; then, $.45\times.45\times.45=.091125$, which is greater than .0837, showing that .45 is too large. So .44 is chosen as a trial value; then, $.44\times.44\times.44=.085184$, which is still too large. The root must therefore be less than .44. So .43 is assumed; then, $.43\times.43\times.43=.079507$, which is smaller than .0837, indicating that the root must be greater than .43. Therefore, it is known that the root lies between .43 and .44. Let .437 be tried; then, $.437\times.437\times.437=.083453453$, or .0835, very nearly. This is so close to .0837 that, for all practical purposes, it may be said that $\sqrt[3]{.0837}=.437$. Ans.

Example 4.—Find the value of $\sqrt[3]{442{,}800}$ to four significant figures by trial and error.

Solution.—When pointed off, the number becomes 442′800. The first figure of the root must be 7, because $7^3=343$, which is less than 442, the first period, whereas $8^3=512$, which is larger than 442. As the given number has two periods, the root will have two figures preceding the decimal point. Let 76 be chosen for the first trial; then, $76\times76\times76 = 438{,}976$, which is less than 442,800, indicating that the root must be greater than 76. If 77 is tried, it is found to be too large, because $77^3 = 456{,}533$, which exceeds 442,800. Therefore, the root lies somewhere between 76 and 77. As 438,976 is closer to 442,800 than is 456,533, the root is closer to 76 than to 77. So, let 76.3 be chosen as a trial value; then, $76.3^3 = 444{,}194.947$, which is larger than 442,800, indicating that 76.3 is too large. If 76.2 is tried, it is found that $76.2^3 = 442{,}450.728$, which is a trifle smaller than 442,800, but very close to it. It is concluded, then, that the root is between 76.2 and 76.3, but not much greater than 76.2. Take 76.21 as a trial value, and $76.21^3 = 442{,}625$, very nearly. This is still less than the given number, so 76.22 is tried as the root. Then, $76.22^3 = 442{,}799$, which is so nearly equal to 442,800 that 76.22 may be considered as the required root, to four significant figures. Ans.

13. Roots of Numbers Containing Fractions.—If the number is a fraction or if it contains a fraction, the simplest way of treating it is to reduce the fraction to a decimal and then proceed as in the preceding examples. For example, suppose that the cube root of $\frac{3}{8}$ is to be found by trial and error. Reduce the fraction to a decimal, thus: $\frac{3}{8} = 3 \div 8 = .375$. Then, the cube root of .375 is found by the method already explained, and the result thus obtained is the root of $\frac{3}{8}$. If the cube root of a number like $17\frac{26}{47}$ is to be found, the fractional part is reduced to a decimal; thus, $\frac{26}{47} = 26 \div 47 = .5532$. The entire number then becomes 17.5532, and the root is found in the manner already explained. The following examples will serve to make the foregoing instructions clearer:

Example 1.—Find by trial and error the cube root of $\frac{5}{8}$, to three significant figures.

Solution.—The fraction $\frac{5}{8}$, reduced to a decimal, becomes .625. As .625 is wholly a decimal the cube root will be wholly a decimal. The first period of .625 contains three figures, the remaining periods consisting of ciphers; that is, the number when pointed off may be written .625′000′000. The largest number whose cube is less than .625 is .8, because $.8^3 = .512$, whereas $.9^3 = .729$. Therefore, the first figure of the root is 8 and the root lies somewhere between .8 and .9. Choose .85 as a trial root; then, $.85^3 = .614125$, which is less than .625. So, try .86 as the root; then, $.86^3 = .636056$, which is too large, as it exceeds .625. The root of .625 therefore lies between .85 and .86. Choose .855 as a trial root; then, $.855^3 = .625026375$, which is so close to .625 that no further calculations are necessary, and .855 may be taken as the cube root of $\frac{5}{8}$=.625. Ans.

EXAMPLE 2.—Find the value of $\sqrt[3]{470\frac{3}{4}}$ to three significant figures by trial and error.

SOLUTION.—First reduce $\frac{3}{4}$ to a decimal; thus, $\frac{3}{4}=3\div4=.75$. The number whose cube root is to be found is then 470.75. When this number is pointed off, it becomes 470.750, showing that the root has one figure before the decimal point. The largest number whose cube is less than 470 is 7, because $7^3=343$, whereas $8^3=512$. So the first figure of the root is 7 and the required root is between 7 and 8. Let 7.5 be assumed; then, $7.5^3=421.875$, which is less than 470.75. So 7.6 is tried; then, $7.6^3=438.976$, which is still too small. Next, 7.7 is tried; then, $7.7^3=456.533$, which also is less than 470.75. Therefore, 7.8 is tried; then, $7.8^3=474.552$, which is too large. It is now seen that the root is greater than 7.7 but less than 7.8. As the cube of 7.8 is 474.552, which is close to the given number 470.75, it is plain that the root must be close to 7.8, but less than 7.8. So, 7.78 is tried; then, $7.78^3=470.911$, which is so close to 470.75 that no further trials are necessary, and the required root is taken as 7.78. Ans.

EXAMPLES FOR PRACTICE

Find the cube roots of the following numbers, by the trial-and-error method, to three significant figures:

		Ans.
(*a*)	18,610	(*a*) 26.5
(*b*)	.3505	(*b*) .705
(*c*)	160,000,000	(*c*) 543
(*d*)	54.5	(*d*) 3.79
(*e*)	$\frac{1}{2}$	(*e*) .794
(*f*)	$12\frac{5}{16}$	(*f*) 2.31

TABLE METHOD

14. In Table I are given the cubes of all whole numbers from 1 to 100, inclusive; likewise, the table gives the cube roots of all numbers between 1 and 1,000,000 that have exact roots. By means of this table it is possible to find directly the first two significant figures in the cube root of any number and to find the third significant figure by a simple calculation. It will be observed that the numbers in the table are all whole numbers, and that none contains more than seven figures. Therefore, to use the table for finding cube roots, the numbers whose roots are to be found must be altered in such a way as to bring them within the range of the table. The method of doing this will now be described.

15. As already explained, digits only are considered in determining the significant figures of a number; therefore, 5, 50, .5, and .05 have the same significant figure, 5. The cubes of these numbers are 125, 125,000, .125, and .000125, respectively, and thus it will be seen that these cubes also have the same significant figures, 1, 2, and 5, and the same significant part, 125. The numbers 125, 125,000, .125, and .000125, if pointed off preparatory to extracting the cube roots, become 125, 125′000, .125, and .000′125. It will be seen at once that each of these contains the same significant part, 125, and that these three significant figures form a single period in each number. The only difference among the numbers is in the position of the decimal point. Consequently, their cube roots will have the same significant figure, 5, but the position of the decimal point will be different. Thus, as 125 is a whole number of one period, its root is a single figure, 5; as 125′000 has two periods, its root has two figures, 50; as .125 is a decimal and comprises one period, its root is a decimal of one figure, or .5; and as .000′125 is a two-period decimal, its root is a decimal of two figures, or .05, because, when a decimal begins with ciphers, there is a cipher in the root, following the decimal point, for each complete period of *three* ciphers directly following the decimal point in the number.

16. The principle explained in the preceding article is important and should be thoroughly understood, as it forms the basis for the use of Table I for finding cube root. Suppose, for example, that the cube root of .004096 is to be found. The table contains no decimals whatever. But, if .004096 is pointed off, it becomes .004′096, which is a number of two periods with a single figure, 4, in the first period. The significant part of the number is 4096. In the table can be found 4,096, which also contains two periods, with the figure 4 alone in the first period, and which has the same significant figures. Therefore, 4,096 and .004096 will have the same significant figures in their roots, the only difference being in the location of the decimal point. The cube root of 4,096 is 16, according to the table. The cube root of .004096 contains these same significant figures, 16, but as .004′096 has two periods and is wholly a decimal, its root must have two figures and be wholly a decimal. Therefore, the cube root of .004096 must be .16.

17. It has been shown that after a number has been pointed off into full periods of three figures each, by the method explained in Arts. **8** to **11,** the position of the decimal point does not affect the figures of the root. Therefore, after the periods have been determined the number may be considered as a whole number while the figures of the root are being found. If this whole number does not consist of two periods it must be altered, because no number having more than two periods is given in the table and because accurate results cannot usually be obtained by using only one period.

If the number has only one period, three zeros are annexed to form the second period. Thus, if 3 is the number whose cube root is to be found, it is necessary to annex three ciphers to form the second period; the altered number is 3′000. If the number contains more than two periods, ignore the periods after the second, unless the first figure of the third period is 5 or more, in which case increase the last figure in the second period by 1. If the number were 2,743,879, it would be pointed off thus, 2′743′879. As there are three periods, the last period is dropped. Since the first figure of the third period exceeds 5, the last figure of the second period is increased by 1. The altered number is 2744, and is found in the table.

18. If the cube root of the altered number may be found from the table directly it is merely necessary to locate the decimal point in accordance with its position in the original number; the cube root of the original number has the same significant figures as the cube root of the altered number.

It frequently happens that the altered number cannot be found in Table I. In such a case the numbers next greater and smaller than the altered number are found, and then, by the method to be illustrated in subsequent examples, the root is determined.

19. If the number whose root is to be found contains a fraction, reduce the fraction to an equivalent decimal, and then proceed as already explained. For example, if the cube root of $\frac{11}{16}$ is to be found, the fraction is reduced to a decimal; thus, $\frac{11}{16} = 11 \div 16 = .6875$. This decimal, when pointed off, becomes .687′500, showing that the root is wholly a decimal, with a digit immediately following the decimal point. The significant figures of the root are exactly the same as those of the cube root of 687,500, which becomes the altered number. If the cube root of $37\frac{1}{8}$ is to be found, the fraction $\frac{1}{8}$ is reduced to a decimal, becoming .125, and the number thus becomes 37.125. The altered number is 37,125, which has the same significant figures for its cube root as has 37.125.

EXAMPLE 1.—Find the value of $\sqrt[3]{12{,}817}$ by the use of Table I.

SOLUTION.—The given number, 12,817, contains two periods and is a whole number, so that it need not be altered. The number 12,817 does not appear in the table, but the numbers 12,167 and 13,824 appear, and 12,817 lies somewhere between them. The cube root of 12,167 is 23 and the cube root of 13,824 is 24; therefore, the cube root of 12,817 must lie between 23 and 24. It may be assumed that the desired root is at the same point between 23 and 24 as 12,817 is between 12,167 and 13,824. To determine just what this intermediate value of the root may be, the following simple calculation is made: The difference between the two numbers in the table is found, and is called the **first difference**; thus, $13{,}824 - 12{,}167 = 1{,}657$, which is the first difference. Next the difference between the given number and the *smaller* number in the table is found, and is called the **second difference**; thus, $12{,}817 - 12{,}167 = 650$, which is the second difference. The second difference is then divided by the first difference; thus, $650 \div 1{,}657 = .39$, or .4, nearly. This figure then becomes the third figure of the root. As the first two figures were already found to be 2 and 3, because the root is greater than 23 but less than 24, the required root is 23.4. Ans.

PROOF.—The accuracy of any result found in extracting cube root may be tested very readily by simply cubing the value found for the root. The product should be approximately equal to the given number. In the preceding example, for instance, the root was found to be 23.4. By the foregoing test, $23.4^3 = 23.4 \times 23.4 \times 23.4 = 12{,}813$, very nearly. As this is very close to 12,817, the given number, it is concluded that 23.4 is the cube root, correct to three significant figures.

Example 2.—Find by the table method the cube root of .0732 to three decimal places.

Solution.—When pointed off, the given number becomes .073′200. The significant figures of this number are the same as if the number were 73,200; also, the significant figures of the cube root of .073200 are the same as those of the cube root of 73,200, because each contains two periods composed of like figures. The only difference is that the cube root of .073200 is wholly a decimal, whereas the cube root of 73,200 is a mixed number in which the decimal point occurs after the second figure. All that is necessary, therefore, is to find the cube root of 73,200 from Table I and then to shift the decimal point in that root until the root is wholly a decimal. The result will then be the cube root of .0732. In the table, 73,200 lies between 68,921 and 74,088, the cube roots of which are 41 and 42, respectively. Therefore, the cube root of 73,200 is greater than 41 and less than 42. The third significant figure of the root is found as follows:

74,088		73,200	
68,921		68,921	
5,167	*first difference*	4,279	*second difference*

Second difference÷*first difference*=4,279÷5,167=.8

Then, the third significant figure is 8 and the cube root of 73,200 is 41.8. The cube root of .0732 has exactly the same significant figures, but as .0732 is wholly a decimal, its cube root is wholly a decimal; therefore, $\sqrt[3]{.0732}=.418$. Ans.

Example 3.—What is the cube root of .18526?

Solution.—When pointed off, the given number becomes .185′260. The cube root of this number has the same significant figures as the cube root of 185,260, but is wholly a decimal. In the table, 185,260 lies between 185,193 and 195,112, showing that the cube root of 185,260 lies between 57 and 58. The first difference is 195,112−185,193=9,919 and the second difference is 185,260−185,193=67; then, 67÷9,919=.007, nearly. Annex this to 57, and the root is 57.007 or 57.01 to four significant figures. As the cube root of .18526 has the same significant figures, but is wholly a decimal, it must be .5701. Ans.

Example 4.—Find from the table the value of $\sqrt[3]{293,475,910}$.

Solution.—When pointed off into periods, the given number becomes 293′475′910. The third period is dropped; but, as it begins with 9, the last figure of the second period is increased by 1, as explained in Art. **17.** The altered number thus becomes 293′476. This is not found in the table, but 287,496 and 300,763 are found there. Therefore the required root is greater than 66 and less than 67. The first difference is 300,763−287,496=13,267 and the second difference is 293,476−287,496=5,980. Then, 5,980÷13,267=.45, showing that the third and fourth significant figures are 4 and 5. The cube root of 287,496 is 66 according to the table, and so the root of 293,476 is 66.45. The original number, however, is a whole number containing three periods; therefore, its root must have three figures ahead of the decimal point. Hence, the root is 664.5. Ans.

Example 5.—What is the cube root of .0007425?

Solution.—The given number, pointed off into complete periods, is .000′742′500. According to Art. **17**, this number must be changed to one containing the same significant figures and having two periods. The first period, consisting of ciphers, is therefore dropped, and the altered number is 742,500. It does not appear in the table, but 729,000 and 753,571, the cubes, respectively, of 90 and 91, are given therein. The first difference is 753,571—729,000=24,571, and the second difference is 742,500—729,000=13,500. Then, 13,500÷24,571=.5, approximately. The cube root of 729,000 is 90, and to this must be annexed the .5 just found, giving 90.5 as the approximate cube root of 742,500. The original number, .0007425, has one complete period of ciphers following the decimal point, and so the root must be a decimal having one cipher following the decimal point and in front of the first significant figure: Therefore, the required root is .0905. Ans.

Example 6.—Find the value of $\sqrt[3]{\frac{273}{1054}}$.

Solution.—First reduce the fraction to a decimal; thus, $\frac{273}{1054}$=273 ÷1,054=.259013. The problem therefore is to find the cube root of .259013. This number, when pointed off, is .259′013, and the altered number is 259,013. This altered number is not in Table I, but it lies between 250,047 and 262,144, the cube roots of which are 63 and 64, respectively. The first difference is 262,144—250,047=12,097, and the second difference is 259,013—250,047=8,966. Then, 8,966÷12,097=.7, and the cube root of 259,013 is 63.7. The cube root of .259013 has the same significant figures, but is wholly a decimal; therefore, it must be .637. Ans.

Example 7.—What is the cube root of 3$\frac{2}{3}$?

Solution.—When reduced to a decimal, $\frac{2}{3}$ becomes .6666+ or .667, and the number whose root is to be found is then 3.667. The altered number is 3,667, which lies between 3,375 and 4,096 in the table; and as the roots of 3,375 and 4,096 are 15 and 16, respectively, the root of 3,667 must lie between 15 and 16. The first difference is 4,096—3,375 =721, and the second difference is 3,667—3,375=292. Then, 292÷721 =.4. So the cube root of 3,667 is 15.4. The cube root of 3.667, which has one period preceding the decimal point, has the same significant figures, with one figure preceding the decimal point; therefore, the cube root required is 1.54. Ans.

EXAMPLES FOR PRACTICE

By the use of Table I find the cube roots of the following numbers to three significant figures:

(*a*)	860,000	Ans.	(*a*)	95.1
(*b*)	27.27		(*b*)	3.01
(*c*)	.09125		(*c*)	.450
(*d*)	127.4		(*d*)	5.03
(*e*)	$56\frac{7}{8}$		(*e*)	3.85
(*f*)	$\frac{49}{64}$		(*f*)	.915

EXACT METHOD

20. If the cube root of a number is to be found accurately to more than three significant figures, neither of the preceding methods can be used successfully. In such a case, an exact method is required. There are a number of such exact methods, but the one about to be described is simple and may be used for finding the cube root of any number. It is most easily explained by the solution of an example.

21. Cube Roots of Whole Numbers.—Suppose that it is desired to find the cube root of 11,390,625. First separate the number into periods of three figures each, commencing at the right. This gives 11′390′625. Draw a line at the right of the number and find the largest number whose cube is not greater than the left-hand period. Place this number as the first figure of the answer.

The left-hand period is 11 and the largest number whose cube will not be greater than 11 is 2, since $3^3 = 27$.

Place the 2 as the first figure of the answer and subtract the cube of this number from the left-hand period of the number. To the remainder thus obtained annex the next period of the number. The process thus far is as follows:

$$\begin{array}{l} 11'390'625(2 \\ 8 \\ \hline 3390 \end{array}$$

It is next necessary to find a *trial divisor*. This is obtained by considering the number in the answer as tens, *squaring* it, and multiplying the result by 3. 2 considered as tens is 20, which squared gives 400. 400 multiplied by 3 gives 1,200 as a trial divisor.

The trial divisor is contained 2 times in the remainder 3,390. Place the 2 as the second figure of the answer. This gives the following:

$$\begin{array}{rl}
 & 1\,1'3\,9\,0'6\,2\,5\,(\,2\,2 \\
 & \,8 \\
3\times20^2=1200 & |\ \overline{3}\,3\,9\,0
\end{array}$$

In order to obtain a *complete divisor*, it is necessary to add together three products.

The *first* product is that obtained for the trial divisor, or, in this case, 1,200.

The *second* product is obtained by considering the first figure of the answer as *tens,* and multiplying it by the *second* figure of the answer, and by 3. In the example being worked, the first figure of the answer considered as tens gives 20, which multiplied by the second figure of the answer, or 2, gives 40, which result multiplied by 3 gives 120 as the second product.

The *third* product is obtained by *squaring* the second figure of the answer, which in this case is $2^2=4$.

Add these three products together and the complete divisor becomes $1,200+120+4=1,324$. The example worked to this point is as follows:

$$\begin{array}{r|l}
 & 1\,1'3\,9\,0'6\,2\,5\,(\,2\,2 \\
 & \,8 \\
3\times20^2=1\,2\,0\,0 & \overline{3}\,3\,9\,0 \\
3\times(20\times2)\ =\ \ 1\,2\,0 & \\
2^2=\qquad\ \ 4 & \\
\cline{1-1}
1\,3\,2\,4 &
\end{array}$$

Next multiply the complete divisor 1,324 by the second figure of the answer and subtract the result from the remainder previously obtained. To the result annex the next period for a *new remainder*. The process is as follows:

$$
\begin{array}{rr|l}
 & & \multicolumn{1}{l}{11'390'625(22} \\
 & & \multicolumn{1}{l}{8} \\
\cline{3-3}
3\times 20^2 = & 1200 & 3390 \\
3\times(20\times 2) = & 120 & \\
2^2 = & 4 & \\
\cline{2-2}
 & 1324 & 2648 \\
\cline{3-3}
 & & 742625
\end{array}
$$

The further processes are but repetitions of those previously described and will not need much explanation.

It is next necessary to find a trial divisor for the remainder. This is done in the same manner as in the first case, the number in the answer being considered as tens, and squared, after which it is multiplied by 3. This will be $220^2 = 48{,}400$, which, multiplied by 3, gives 145,200 for a trial divisor.

The trial divisor divided into the remainder gives 5 as the next figure of the answer.

Next, find the complete divisor by adding together the three products, taking the trial divisor as the first product, 3 times the number already in the answer considered as tens multiplied by the next figure of the answer for the second product, and the square of the third figure of the answer for the third product. Multiply the complete divisor by the third figure of the answer and subtract the result thus obtained from the remainder previously obtained. The complete process is:

$$
\begin{array}{rr|l}
 & & \multicolumn{1}{l}{11'390'625(225 \quad \text{Ans.}} \\
 & & \multicolumn{1}{l}{8} \\
\cline{3-3}
3\times 20^2 = & 1200 & 3390 \\
3\times(20\times 2) = & 120 & \\
2^2 = & 4 & \\
\cline{2-2}
 & 1324 & 2648 \\
\cline{3-3}
3\times 220^2 = & 145200 & 742625 \\
3\times(220\times 5) = & 3300 & \\
5^2 = & 25 & \\
\cline{2-2}
 & 148525 & 742625 \\
\cline{3-3}
\end{array}
$$

Since the product obtained by multiplying the complete divisor by the third term of the answer is just equal to the remainder, the process is complete, and the answer, or 225, is the cube root of the number 11,390,625.

22. Cube Roots of Decimals.—In extracting the cube root of a decimal, proceed as with a whole number, taking care that each period contains *three* figures. Begin the pointing off at the decimal point, going toward the right. If the last period does not contain three figures, annex ciphers until it does.

The number of decimal places in the answer will equal the number of periods in the decimal whose root is to be extracted.

Example.—What is the cube root of .009129329?

Solution.—

		.009'129'329 (.209 Ans.
		8
$3\times20^2=$	1200	1129
		0000
$3\times200^2=$	120000	1129329
$3\times(200\times9)=$	5400	
$9^2=$	81	
	125481	1129329

In this particular example it will be noticed that in the first case the trial divisor 1,200 is greater than the remainder 1,129; consequently, the next figure of the answer is 0 and it is necessary to bring down the next period for a new remainder, after which the regular process is followed.

23. Roots of Mixed Numbers.—One example of extracting the cube root of a mixed number will be given here.

Example.—What is the cube root of 47.832147?

Solution.—

		47.832'147 (3.63 Ans.
		27
$3\times30^2=$	2700	20832
$3\times(30\times6)=$	540	
$6^2=$	36	
	3276	19656
$3\times360^2=$	388800	1176147
$3\times(360\times3)=$	3240	
$3^2=$	9	
	392049	1176147

24. Rule for Cube Root by Exact Method.—The series of operations that must be performed in order to find the cube root of a number by the method described in the preceding examples may be stated in the form of a rule, as follows:

Rule.—*Separate the number into periods of three figures each, commencing at the right; if the number contains a decimal, begin at the decimal point and work both ways.*

Find the largest number whose cube is not greater than the left-hand period and write this number as the first figure of the answer.

Subtract the cube from the left-hand period and to this result annex the next period of the number.

Square the number in the answer considered as tens, and multiply this result by 3.

Use the result thus obtained for a trial divisor to divide into the remainder, and place the number resulting from this division as the next figure of the answer.

To the trial divisor, add the result obtained by multiplying the first figure of the answer considered as tens by the second figure of the answer and by 3; also add the square of the second figure. The sum thus obtained is the complete divisor.

Multiply the complete divisor by the second figure of the answer and subtract the result thus obtained from the remainder.

To this result annex the next period of the number and proceed in the manner described until all the periods have been used.

25. Cube Roots of Fractions.—If the given number is in the form of a fraction, and it is required to find the cube root of it, the simplest and most exact method is to reduce the fraction to a decimal and extract the cube root of the decimal. If, however, the numerator and denominator of the fraction are perfect powers, extract the cube root of each separately, and write the root of the numerator for a new numerator, and the root of the denominator for a new denominator.

Rule.—*Extract the cube root of the numerator and denominator separately if each is a perfect cube; otherwise, reduce the fraction to a decimal, and extract the root of the decimal.*

EXAMPLE 1.—What is the cube root of $\frac{27}{64}$?

SOLUTION.— $\sqrt[3]{\frac{27}{64}}=\frac{\sqrt[3]{27}}{\sqrt[3]{64}}=\frac{3}{4}$. Ans.

EXAMPLE 2.—What is the cube root of $\frac{1}{4}$?

SOLUTION.—Since $\frac{1}{4}=.25$, $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{.25}=.62996$. Ans.

26. Memorizing the Exact Method.—The extraction of cube root is an operation with which many students have difficulty, particularly when the exact method is used. The trouble is not in performing the several calculations, but in remembering the order in which they must be done and in finding the trial divisor and the complete divisor. Reference to the solutions given in Arts. **21** to **23** shows that, after cubing the first figure of the root and obtaining the first remainder, the same series of operations is repeated until the work is completed. This series of operations is made up of the following steps:

1. Finding the trial divisor.
2. Finding the next figure of the root.
3. Finding the complete divisor.
4. Multiplying the complete divisor by the last figure of the root thus far found.
5. Subtracting and forming a new dividend.

27. The only steps that are likely to cause any trouble are the first and the third, or finding the trial divisor and the complete divisor; but these may easily be remembered by fixing in mind the following expression:

$$3a^2+3ab+b^2$$

The first term, or $3a^2$, represents the trial divisor, and the whole expression, $3\,a^2+3\,a\,b+b^2$ represents the complete divisor. To indicate just how these terms are used to represent the trial and complete divisors, consider again the solution of the example in Art. **21**, which is repeated here for convenience.

$$
\begin{array}{lrr|l}
 & & & 11'390'625(225\quad \text{Ans.} \\
 & & & \underline{8} \\
\textit{trial divisor} \ =3a^2= & 3\times 20^2= & 1200 & 3390 \\
3ab=3\times(20\times 2)= & & 120 & \\
b^2= & 2^2= & \underline{4} & \\
\textit{complete divisor} \ =3a^2+3ab+b^2= & & 1324 & \underline{2648} \\
 & & & 742625 \\
\textit{trial divisor} \ =3a^2= & 3\times 220^2= & 145200 & \\
3ab=3\times(220\times 5)= & & 3300 & \\
b^2= & 5^2= & \underline{25} & \\
\textit{complete divisor} \ =3a^2+3ab+b^2= & & 148525 & \underline{742625}
\end{array}
$$

28. After the first figure of the root has been cubed and its cube has been subtracted from the first period, the next period is brought down, giving 3,390 as the first dividend, for which a trial divisor must be found. Let a represent 10 times the root *thus far found,* which in this case makes $a=10\times 2=20$. Then, the trial divisor can easily be calculated, as it is $3a^2$. As $a=20$, it is readily seen that $3a^2=3\times 20^2=3\times 20\times 20=1{,}200$. This trial divisor, 1,200, is contained 2 times in the dividend 3,390; so 2 is the next figure of the root, and is written down there. Now let b represent the last figure of the root *thus far found,* which in this case is the second 2; then, $b=2$. To obtain the complete divisor, add $3ab$ and b^2 to the trial divisor $3a^2$, as shown. As $a=20$ and $b=2$, it follows that $3ab=3\times(a\times b)=3\times(20\times 2)=3\times 40=120$; and $b^2=2^2=2\times 2=4$. Then, $3a^2+3ab+b^2=1{,}200+120+4=1{,}324$, the first complete divisor. Multiply 1,324 by 2, subtract the product, 2,648, from the first dividend, and bring down the next period, producing the second dividend, or 742,625.

29. The next operation is similar in every way to that just performed. First, a trial divisor must be found. The figures of the root *thus far found* are 22, and a represents 10 times this value; that is, $a=10\times 22=220$. Then, the trial divisor is $3a^2=3\times 220^2=3\times 220\times 220=145{,}200$, as shown. This is contained 5 times in the second dividend, and so 5 is written as the next figure of the root. Then, $b=5$, because b

always represents the last figure of the root *thus far found.* Therefore, $3ab = 3\times(a\times b) = 3\times(220\times5) = 3\times1.100 = 3.300$: and $b^2 = 5^2 = 25$. The second complete divisor is then $3a^2 + 3ab + b^2 = 145{,}200 + 3{,}300 + 25 = 148{,}525$, as shown. This is multiplied by the last figure of the root, 5, and the product is written under the second dividend. As there is no remainder, the work is ended and the required cube root is 225.

30. To assist further in fixing in mind the steps just explained, the following restatement of principles is made:

1. The trial divisor at any stage of the operation of finding cube root is always equal to $3a^2$, in which a is 10 times the value of the root *thus far found.*

2. The complete divisor is the sum of the trial divisor, $3a^2$, and the products $3ab$ and b^2, in which b is the value of the last figure of the root *thus far found;* that is, the complete divisor is $3a^2 + 3ab + b^2$.

If this expression is committed to memory, together with an understanding of what a and b stand for, it should be easy to find the cube root of any number under any circumstances.

EXAMPLES FOR PRACTICE

Find the cube roots of the following numbers by the exact method:

		Ans.	
(*a*)	1,860,867	(*a*)	123
(*b*)	.185193	(*b*)	.57
(*c*)	636,056	(*c*)	86
(*d*)	87,528.384	(*d*)	44.4
(*e*)	74,088	(*e*)	42
(*f*)	257,259,456	(*f*)	636
(*g*)	$\frac{125}{4096}$	(*g*)	$\frac{5}{16}$
(*h*)	$\frac{2197}{27000}$	(*h*)	$\frac{13}{30}$
(*i*)	$\frac{87}{125}$	(*i*)	.8862
(*j*)	$\frac{3}{8}$	(*j*)	.7211

APPLICATIONS OF CUBE ROOT

31. Diameter of a Sphere.—There are many sorts of problems whose solutions require the extraction of cube root. One of the most common cases in which cube root must be extracted is in finding the diameter of a sphere. The rule for finding the diameter is as follows:

Rule.—*To find the diameter of a sphere, extract the cube root of the volume and multiply the result by 1.24.*

If the volume is in cubic inches, the diameter will be in inches; if the volume is in cubic feet, the diameter will be in feet; and so on.

Example 1.—A spherical ball contains 2,000 cubic inches. What is the diameter of the ball?

Solution.—According to the rule, the cube root of the volume, or 2,000 cu. in., must first be found. The cube root of 2,000 may be found by any one of the methods previously described. In this case, however, the table method will be used. The number 2,000 does not appear in Table I, but 1,728 and 2,197 are given, their roots being 12 and 13, respectively. Therefore, the cube root of 2,000 is between 12 and 13. The first difference is $2{,}197-1{,}728=469$ and the second difference is $2{,}000-1{,}728=272$. Then, $272\div469=.6$. Consequently, the cube root of 2,000 is 12.6. To find the diameter, the cube root must be multiplied by 1.24. Therefore, the diameter of the ball is

$12.6\times1.24=15.624$, or $15\frac{5}{8}$ in., nearly. Ans.

Example 2.—If a tank in the shape of a sphere must hold 48,000 cubic feet, what must be its inside diameter?

Solution.—First find the cube root of 48,000, using the table method. This value lies between 46,656 and 50,653 in the table. The first difference is $50{,}653-46{,}656=3{,}997$, and the second difference is $48{,}000-46{,}656=1{,}344$. Then, $1{,}344\div3{,}997=.336$, or .34, approximately. The cube root of 46,656 is 36 and so the cube root of 48,000 is 36.34. The diameter of the tank is 1.24 times the root thus found, or

$36.34\times1.24=45$ ft., very closely. Ans.

32. Speed and Coal Consumption of a Steamship. The speed of a steam-driven ship may be increased by forcing the fires, that is, by burning coal more rapidly under the boilers. The amount of coal burned per day, called the coal consumption, is proportional to the cube of the speed. Therefore, if the coal consumption is changed, and it is desired to find what the speed will be at the new rate at which the coal is burned, the following rule may be used:

Rule.—*To find approximately the new speed for a new rate of coal consumption, multiply the new coal consumption by the cube of the known speed, divide the product by the coal consumption corresponding to the known speed, and extract the cube root of the quotient.*

The speeds of vessels are usually expressed in knots, a knot being a speed of 1.15156 miles per hour.

Example 1.—A steamer consumes 100 tons of coal a day when running at a speed of 10 knots. What will be the speed if the coal consumption is increased to 200 tons a day?

Solution.—The new rate of coal consumption is 200 tons a day and the known speed is 10 knots. According to the rule, then, 200 is to be multiplied by the cube of 10; thus, $200 \times 10^3 = 200 \times 1{,}000 = 200{,}000$. This product is then to be divided by the coal consumption at the known speed, or 100 tons a day. Thus, $200{,}000 \div 100 = 2{,}000$. The cube root of this quotient, 2,000, is then the new speed; that is, the new speed is equal to $\sqrt[3]{2{,}000}$, which, by the table method or by either of the other methods, is 12.6. Therefore, the new speed will be 12.6 knots. Ans.

Example 2.—If a steamer consumes 60 tons of coal a day to produce a speed of 9 knots, how many knots would she make if the coal consumption were decreased to 48 tons a day?

Solution.—The known speed is 9 knots, the coal consumption at that speed is 60 tons a day, and the new coal consumption is 48 tons a day. According to the rule, therefore, the new speed is

$$\sqrt[3]{\frac{48 \times 9^3}{60}} = \sqrt[3]{\frac{48 \times 729}{60}} = \sqrt[3]{583.2} = 8.35 \text{ knots. Ans.}$$

TABLE

OF

COMMON LOGARITHMS

OF NUMBERS

From 1 to 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
0	— ∞	**20**	30 103	**40**	60 206	**60**	77 815	**80**	90 309
1	00 000	21	32 222	41	61 278	61	78 533	81	90 849
2	30 103	22	34 242	42	62 325	62	79 239	82	91 381
3	47 712	23	36 173	43	63 347	63	79 934	83	91 908
4	60 206	24	38 021	44	64 345	64	80 618	84	92 428
5	69 897	25	39 794	45	65 321	65	81 291	85	92 942
6	77 815	26	41 497	46	66 276	66	81 954	86	93 450
7	84 510	27	43 136	47	67 210	67	82 607	87	93 952
8	90 309	28	44 716	48	68 124	68	83 251	88	94 448
9	95 424	29	46 240	49	69 020	69	83 885	89	94 939
10	00 000	**30**	47 712	**50**	69 897	**70**	84 510	**90**	95 424
11	04 139	31	49 136	51	70 757	71	85 126	91	95 904
12	07 918	32	50 515	52	71 600	72	85 733	92	96 379
13	11 394	33	51 851	53	72 428	73	86 332	93	96 848
14	14 613	34	53 148	54	73 239	74	86 923	94	97 313
15	17 609	35	54 407	55	74 036	75	87 506	95	97 772
16	20 412	36	55 630	56	74 819	76	88 081	96	98 227
17	23 045	37	56 820	57	75 587	77	88 649	97	98 677
18	25 527	38	57 978	58	76 343	78	89 209	98	99 123
19	27 875	39	59 106	59	77 085	79	89 763	99	99 564
20	30 103	**40**	60 206	**60**	77 815	**80**	90 309	**100**	00 000

N.	L. 0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	44	**43**	**42**
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	**40**	**39**
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	**37**	**36**
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	**34**	**33**
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	**31**	**30**
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	29	**28**
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	**26**
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	**23**
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	**21**
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

P. P.

LOGARITHMS

N.	L. 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	22	**21**
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
250	39 794	811	829	846	863	881	898	915	933	950	
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	
252	40 140	157	175	192	209	226	243	261	278	295	
253	312	329	346	364	381	398	415	432	449	466	
254	483	500	518	535	552	569	586	603	620	637	
255	654	671	688	705	722	739	756	773	790	807	18 1 1.8 2 3.6 3 5.4 4 7.2 5 9.0 6 10.8 7 12.6 8 14.4 9 16.2
256	824	841	858	875	892	909	926	943	960	976	
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	
258	41 162	179	196	212	229	246	263	280	296	313	
259	330	347	363	380	397	414	430	447	464	481	
260	497	514	531	547	564	581	597	614	631	647	
261	664	681	697	714	731	747	764	780	797	814	
262	830	847	863	880	896	913	929	946	963	979	
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	
264	42 160	177	193	210	226	243	259	275	292	308	
265	325	341	357	374	390	406	423	439	455	472	17 1 1.7 2 3.4 3 5.1 4 6.8 5 8.5 6 10.2 7 11.9 8 13.6 9 15.3
266	488	504	521	537	553	570	586	602	619	635	
267	651	667	684	700	716	732	749	765	781	797	
268	813	830	846	862	878	894	911	927	943	959	
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	
270	43 136	152	169	185	201	217	233	249	265	281	
271	297	313	329	345	361	377	393	409	425	441	
272	457	473	489	505	521	537	553	569	584	600	
273	616	632	648	664	680	696	712	727	743	759	
274	775	791	807	823	838	854	870	886	902	917	
275	933	949	965	981	996	*012	*028	*044	*059	*075	16 1 1.6 2 3.2 3 4.8 4 6.4 5 8.0 6 9.6 7 11.2 8 12.8 9 14.4
276	44 091	107	122	138	154	170	185	201	217	232	
277	248	264	279	295	311	326	342	358	373	389	
278	404	420	436	451	467	483	498	514	529	545	
279	560	576	592	607	623	638	654	669	685	700	
280	716	731	747	762	778	793	809	824	840	855	
281	871	886	902	917	932	948	963	979	994	*010	
282	45 025	040	056	071	086	102	117	133	148	163	
283	179	194	209	225	240	255	271	286	301	317	
284	332	347	362	378	393	408	423	439	454	469	
285	484	500	515	530	545	561	576	591	606	621	15 1 1.5 2 3.0 3 4.5 4 6.0 5 7.5 6 9.0 7 10.5 8 12.0 9 13.5
286	637	652	667	682	697	712	728	743	758	773	
287	788	803	818	834	849	864	879	894	909	924	
288	939	954	969	984	*000	*015	*030	*045	*060	*075	
289	46 090	105	120	135	150	165	180	195	210	225	
290	240	255	270	285	300	315	330	345	359	374	
291	389	404	419	434	449	464	479	494	509	523	
292	538	553	568	583	598	613	627	642	657	672	
293	687	702	716	731	746	761	776	790	805	820	
294	835	850	864	879	894	909	923	938	953	967	
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	14 1 1.4 2 2.8 3 4.2 4 5.6 5 7.0 6 8.4 7 9.8 8 11.2 9 12.6
296	47 129	144	159	173	188	202	217	232	246	261	
297	276	290	305	319	334	349	363	378	392	407	
298	422	436	451	465	480	494	509	524	538	553	
299	567	582	596	611	625	640	654	669	683	698	
300	712	727	741	756	770	784	799	813	828	842	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
350	54 407	419	432	444	456	469	481	494	506	518	
351	531	543	555	568	580	593	605	617	630	642	
352	654	667	679	691	704	716	728	741	753	765	
353	777	790	802	814	827	839	851	864	876	888	
354	900	913	925	937	949	962	974	986	998	*011	**13** 1 1.3 2 2.6 3 3.9 4 5.2 5 6.5 6 7.8 7 9.1 8 10.4 9 11.7
355	55 023	035	047	060	072	084	096	108	121	133	
356	145	157	169	182	194	206	218	230	242	255	
357	267	279	291	303	315	328	340	352	364	376	
358	388	400	413	425	437	449	461	473	485	497	
359	509	522	534	546	558	570	582	594	606	618	
360	630	642	654	666	678	691	703	715	727	739	
361	751	763	775	787	799	811	823	835	847	859	
362	871	883	895	907	919	931	943	955	967	979	
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098	
364	56 110	122	134	146	158	170	182	194	205	217	
365	229	241	253	265	277	289	301	312	324	336	
366	348	360	372	384	396	407	419	431	443	455	**12** 1 1.2 2 2.4 3 3.6 4 4.8 5 6.0 6 7.2 7 8.4 8 9.6 9 10.8
367	467	478	490	502	514	526	538	549	561	573	
368	585	597	608	620	632	644	656	667	679	691	
369	703	714	726	738	750	761	773	785	797	808	
370	820	832	844	855	867	879	891	902	914	926	
371	937	949	961	972	984	996	*008	*019	*031	*043	
372	57 054	066	078	089	101	113	124	136	148	159	
373	171	183	194	206	217	229	241	252	264	276	
374	287	299	310	322	334	345	357	368	380	392	
375	403	415	426	438	449	461	473	484	496	507	
376	519	530	542	553	565	576	588	600	611	623	
377	634	646	657	669	680	692	703	715	726	738	
378	749	761	772	784	795	807	818	830	841	852	**11** 1 1.1 2 2.2 3 3.3 4 4.4 5 5.5 6 6.6 7 7.7 8 8.8 9 9.9
379	864	875	887	898	910	921	933	944	955	967	
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081	
381	58 092	104	115	127	138	149	161	172	184	195	
382	206	218	229	240	252	263	274	286	297	309	
383	320	331	343	354	365	377	388	399	410	422	
384	433	444	456	467	478	490	501	512	524	535	
385	546	557	569	580	591	602	614	625	636	647	
386	659	670	681	692	704	715	726	737	749	760	
387	771	782	794	805	816	827	838	850	861	872	
388	883	894	906	917	928	939	950	961	973	984	
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095	**10** 1 1.0 2 2.0 3 3.0 4 4.0 5 5.0 6 6.0 7 7.0 8 8.0 9 9.0
390	59 106	118	129	140	151	162	173	184	195	207	
391	218	229	240	251	262	273	284	295	306	318	
392	329	340	351	362	373	384	395	406	417	428	
393	439	450	461	472	483	494	506	517	528	539	
394	550	561	572	583	594	605	616	627	638	649	
395	660	671	682	693	704	715	726	737	748	759	
396	770	780	791	802	813	824	835	846	857	868	
397	879	890	901	912	923	934	945	956	966	977	
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086	
399	60 097	108	119	130	141	152	163	173	184	195	
400	206	217	228	239	249	260	271	282	293	304	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
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P. P.

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P. P.

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451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
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P. P.

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P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
500	69 897	906	914	923	932	940	949	958	966	975	
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
502	70 070	079	088	096	105	114	122	131	140	148	
503	157	165	174	183	191	200	209	217	226	234	
504	243	252	260	269	278	286	295	303	312	321	
505	329	338	346	355	364	372	381	389	398	406	
506	415	424	432	441	449	458	467	475	484	492	
507	501	509	518	526	535	544	552	561	569	578	9
508	586	595	603	612	621	629	638	646	655	663	1 0.9
509	672	680	689	697	706	714	723	731	740	749	2 1.8
510	757	766	774	783	791	800	808	817	825	834	3 2.7
511	842	851	859	868	876	885	893	902	910	919	4 3.6
512	927	935	944	952	961	969	978	986	995	*003	5 4.5
513	71 012	020	029	037	046	054	063	071	079	088	6 5.4
514	096	105	113	122	130	139	147	155	164	172	7 6.3
515	181	189	198	206	214	223	231	240	248	257	8 7.2
516	265	273	282	290	299	307	315	324	332	341	9 8.1
517	349	357	366	374	383	391	399	408	416	425	
518	433	441	450	458	466	475	483	492	500	508	
519	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	
521	684	692	700	709	717	725	734	742	750	759	
522	767	775	784	792	800	809	817	825	834	842	8
523	850	858	867	875	883	892	900	908	917	925	1 0.8
524	933	941	950	958	966	975	983	991	999	*008	2 1.6
525	72 016	024	032	041	049	057	066	074	082	090	3 2.4
526	099	107	115	123	132	140	148	156	165	173	4 3.2
527	181	189	198	206	214	222	230	239	247	255	5 4.0
528	263	272	280	288	296	304	313	321	329	337	6 4.8
529	346	354	362	370	378	387	395	403	411	419	7 5.6
530	428	436	444	452	460	469	477	485	493	501	8 6.4
531	509	518	526	534	542	550	558	567	575	583	9 7.2
532	591	599	607	616	624	632	640	648	656	665	
533	673	681	689	697	705	713	722	730	738	746	
534	754	762	770	779	787	795	803	811	819	827	
535	835	843	852	860	868	876	884	892	900	908	
536	916	925	933	941	949	957	965	973	981	989	
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	7
538	73 078	086	094	102	111	119	127	135	143	151	1 0.7
539	159	167	175	183	191	199	207	215	223	231	2 1.4
540	239	247	255	263	272	280	288	296	304	312	3 2.1
541	320	328	336	344	352	360	368	376	384	392	4 2.8
542	400	408	416	424	432	440	448	456	464	472	5 3.5
543	480	488	496	504	512	520	528	536	544	552	6 4.2
544	560	568	576	584	592	600	608	616	624	632	7 4.9
545	640	648	656	664	672	679	687	695	703	711	8 5.6
546	719	727	735	743	751	759	767	775	783	791	9 6.3
547	799	807	815	823	830	838	846	854	862	870	
548	878	886	894	902	910	918	926	933	941	949	
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N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
550	74 036	044	052	060	068	076	084	092	099	107	
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552	194	202	210	218	225	233	241	249	257	265	
553	273	280	288	296	304	312	320	327	335	343	
554	351	359	367	374	382	390	398	406	414	421	
555	429	437	445	453	461	468	476	484	492	500	
556	507	515	523	531	539	547	554	562	570	578	
557	586	593	601	609	617	624	632	640	648	656	
558	663	671	679	687	695	702	710	718	726	733	
559	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	850	858	865	873	881	889	8 1 0.8 2 1.6 3 2.4 4 3.2 5 4.0 6 4.8 7 5.6 8 6.4 9 7.2
561	896	904	912	920	927	935	943	950	958	966	
562	974	981	989	997	*005	*012	*020	*028	*035	*043	
563	75 051	059	066	074	082	089	097	105	113	120	
564	128	136	143	151	159	166	174	182	189	197	
565	205	213	220	228	236	243	251	259	266	274	
566	282	289	297	305	312	320	328	335	343	351	
567	358	366	374	381	389	397	404	412	420	427	
568	435	442	450	458	465	473	481	488	496	504	
569	511	519	526	534	542	549	557	565	572	580	
570	587	595	603	610	618	626	633	641	648	656	
571	664	671	679	686	694	702	709	717	724	732	
572	740	747	755	762	770	778	785	793	800	808	
573	815	823	831	838	846	853	861	868	876	884	
574	891	899	906	914	921	929	937	944	952	959	
575	967	974	982	989	997	*005	*012	*020	*027	*035	
576	76 042	050	057	065	072	080	087	095	103	110	
577	118	125	133	140	148	155	163	170	178	185	
578	193	200	208	215	223	230	238	245	253	260	
579	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	
581	418	425	433	440	448	455	462	470	477	485	
582	492	500	507	515	522	530	537	545	552	559	
583	567	574	582	589	597	604	612	619	626	634	
584	641	649	656	664	671	678	686	693	701	708	7 1 0.7 2 1.4 3 2.1 4 2.8 5 3.5 6 4.2 7 4.9 8 5.6 9 6.3
585	716	723	730	738	745	753	760	768	775	782	
586	790	797	805	812	819	827	834	842	849	856	
587	864	871	879	886	893	901	908	916	923	930	
588	938	945	953	960	967	975	982	989	997	*004	
589	77 012	019	026	034	041	048	056	063	070	078	
590	085	093	100	107	115	122	129	137	144	151	
591	159	166	173	181	188	195	203	210	217	225	
592	232	240	247	254	262	269	276	283	291	298	
593	305	313	320	327	335	342	349	357	364	371	
594	379	386	393	401	408	415	422	430	437	444	
595	452	459	466	474	481	488	495	503	510	517	
596	525	532	539	546	554	561	568	576	583	590	
597	597	605	612	619	627	634	641	648	656	663	
598	670	677	685	692	699	706	714	721	728	735	
599	743	750	757	764	772	779	786	793	801	808	
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N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N	L. 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
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631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
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P. P.

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9	5.4

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
650	81 291	298	305	311	318	325	331	338	345	351	
651	358	365	371	378	385	391	398	405	411	418	
652	425	431	438	445	451	458	465	471	478	485	
653	491	498	505	511	518	525	531	538	544	551	
654	558	564	571	578	584	591	598	604	611	617	
655	624	631	637	644	651	657	664	671	677	684	
656	690	697	704	710	717	723	730	737	743	750	
657	757	763	770	776	783	790	796	803	809	816	
658	823	829	836	842	849	856	862	869	875	882	
659	889	895	902	908	915	921	928	935	941	948	
660	954	961	968	974	981	987	994	*000	*007	*014	
661	82 020	027	033	040	046	053	060	066	073	079	**7** 1 0.7 2 1.4 3 2.1 4 2.8 5 3.5 6 4.2 7 4.9 8 5.6 9 6.3
662	086	092	099	105	112	119	125	132	138	145	
663	151	158	164	171	178	184	191	197	204	210	
664	217	223	230	236	243	249	256	263	269	276	
665	282	289	295	302	308	315	321	328	334	341	
666	347	354	360	367	373	380	387	393	400	406	
667	413	419	426	432	439	445	452	458	465	471	
668	478	484	491	497	504	510	517	523	530	536	
669	543	549	556	562	569	575	582	588	595	601	
670	607	614	620	627	633	640	646	653	659	666	
671	672	679	685	692	698	705	711	718	724	730	
672	737	743	750	756	763	769	776	782	789	795	
673	802	808	814	821	827	834	840	847	853	860	
674	866	872	879	885	892	898	905	911	918	924	
675	930	937	943	950	956	963	969	975	982	988	
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	
677	83 059	065	072	078	085	091	097	104	110	117	
678	123	129	136	142	149	155	161	168	174	181	
679	187	193	200	206	213	219	225	232	238	245	
680	251	257	264	270	276	283	289	296	302	308	
681	315	321	327	334	340	347	353	359	366	372	
682	378	385	391	398	404	410	417	423	429	436	
683	442	448	455	461	467	474	480	487	493	499	
684	506	512	518	525	531	537	544	550	556	563	**6** 1 0.6 2 1.2 3 1.8 4 2.4 5 3.0 6 3.6 7 4.2 8 4.8 9 5.4
685	569	575	582	588	594	601	607	613	620	626	
686	632	639	645	651	658	664	670	677	683	689	
687	696	702	708	715	721	727	734	740	746	753	
688	759	765	771	778	784	790	797	803	809	816	
689	822	828	835	841	847	853	860	866	872	879	
690	885	891	897	904	910	916	923	929	935	942	
691	948	954	960	967	973	979	985	992	998	*004	
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693	073	080	086	092	098	105	111	117	123	130	
694	136	142	148	155	161	167	173	180	186	192	
695	198	205	211	217	223	230	236	242	248	255	
696	261	267	273	280	286	292	298	305	311	317	
697	323	330	336	342	348	354	361	367	373	379	
698	386	392	398	404	410	417	423	429	435	442	
699	448	454	460	466	473	479	485	491	497	504	
700	510	516	522	528	535	541	547	553	559	566	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

P. P.

LOGARITHMS

N.	L. 0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
800	90 309	314	320	325	331	336	342	347	352	358	
801	363	369	374	380	385	390	396	401	467	412	
802	417	423	428	434	439	445	450	455	461	466	
803	472	477	482	488	493	499	504	509	515	520	
804	526	531	536	542	547	553	558	563	569	574	
805	580	585	590	596	601	607	612	617	623	628	
806	634	639	644	650	655	660	666	671	677	682	
807	687	693	698	703	709	714	720	725	730	736	
808	741	747	752	757	763	768	773	779	784	789	
809	795	800	806	811	816	822	827	832	838	843	
810	849	854	859	865	870	875	881	886	891	897	
811	902	907	913	918	924	929	934	940	945	950	
812	956	961	966	972	977	982	988	993	998	*004	**6**
813	91 009	014	020	025	030	036	041	046	052	057	1 | 0.6 2 | 1.2 3 | 1.8 4 | 2.4 5 | 3.0 6 | 3.6 7 | 4.2 8 | 4.8 9 | 5.4
814	062	068	073	078	084	089	094	100	105	110	
815	116	121	126	132	137	142	148	153	158	164	
816	169	174	180	185	190	196	201	206	212	217	
817	222	228	233	238	243	249	254	259	265	270	
818	275	281	286	291	297	302	307	312	318	323	
819	328	334	339	344	350	355	360	365	371	376	
820	381	387	392	397	403	408	413	418	424	429	
821	434	440	445	450	455	461	466	471	477	482	
822	487	492	498	503	508	514	519	524	529	535	
823	540	545	551	556	561	566	572	577	582	587	
824	593	598	603	609	614	619	624	630	635	640	
825	645	651	656	661	666	672	677	682	687	693	
826	698	703	709	714	719	724	730	735	740	745	
827	751	756	761	766	772	777	782	787	793	798	
828	803	808	814	819	824	829	834	840	845	850	
829	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	950	955	
831	960	965	971	976	981	986	991	997	*002	*007	**5**
832	92 012	018	023	028	033	038	044	049	054	059	1 | 0.5 2 | 1.0 3 | 1.5 4 | 2.0 5 | 2.5 6 | 3.0 7 | 3.5 8 | 4.0 9 | 4.5
833	065	070	075	080	085	091	096	101	106	111	
834	117	122	127	132	137	143	148	153	158	163	
835	169	174	179	184	189	195	200	205	210	215	
836	221	226	231	236	241	247	252	257	262	267	
837	273	278	283	288	293	298	304	309	314	319	
838	324	330	335	340	345	350	355	361	366	371	
839	376	381	387	392	397	402	407	412	418	423	
840	428	433	438	443	449	454	459	464	469	474	
841	480	485	490	495	500	505	511	516	521	526	
842	531	536	542	547	552	557	562	567	572	578	
843	583	588	593	598	603	609	614	619	624	629	
844	634	639	645	650	655	660	665	670	675	681	
845	686	691	696	701	706	711	716	722	727	732	
846	737	742	747	752	758	763	768	773	778	783	
847	788	793	799	804	809	814	819	824	829	834	
848	840	845	850	855	860	865	870	875	881	886	
849	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
850	92 942	947	952	957	962	967	973	978	983	988	
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
852	93 044	049	054	059	064	069	075	080	085	090	
853	095	100	105	110	115	120	125	131	136	141	
854	146	151	156	161	166	171	176	181	186	192	
855	197	202	207	212	217	222	227	232	237	242	
856	247	252	258	263	268	273	278	283	288	293	
857	298	303	308	313	318	323	328	334	339	344	6 1 0.6 2 1.2 3 1.8 4 2.4 5 3.0 6 3.6 7 4.2 8 4.8 9 5.4
858	349	354	359	364	369	374	379	384	389	394	
859	399	404	409	414	420	425	430	435	440	445	
860	450	455	460	465	470	475	480	485	490	495	
861	500	505	510	515	520	526	531	536	541	546	
862	551	556	561	566	571	576	581	586	591	596	
863	601	606	611	616	621	626	631	636	641	646	
864	651	656	661	666	671	676	682	687	692	697	
865	702	707	712	717	722	727	732	737	742	747	
866	752	757	762	767	772	777	782	787	792	797	
867	802	807	812	817	822	827	832	837	842	847	
868	852	857	862	867	872	877	882	887	892	897	
869	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
871	94 002	007	012	017	022	027	032	037	042	047	
872	052	057	062	067	072	077	082	086	091	096	5 1 0.5 2 1.0 3 1.5 4 2.0 5 2.5 6 3.0 7 3.5 8 4.0 9 4.5
873	101	106	111	116	121	126	131	136	141	146	
874	151	156	161	166	171	176	181	186	191	196	
875	201	206	211	216	221	226	231	236	240	245	
876	250	255	260	265	270	275	280	285	290	295	
877	300	305	310	315	320	325	330	335	340	345	
878	349	354	359	364	369	374	379	384	389	394	
879	399	404	409	414	419	424	429	433	438	443	
880	448	453	458	463	468	473	478	483	488	493	
881	498	503	507	512	517	522	527	532	537	542	
882	547	552	557	562	567	571	576	581	586	591	
883	596	601	606	611	616	621	626	630	635	640	
884	645	650	655	660	665	670	675	680	685	689	
885	694	699	704	709	714	719	724	729	734	738	
886	743	748	753	758	763	768	773	778	783	787	
887	792	797	802	807	812	817	822	827	832	836	4 1 0.4 2 0.8 3 1.2 4 1.6 5 2.0 6 2.4 7 2.8 8 3.2 9 3.6
888	841	846	851	856	861	866	871	876	880	885	
889	890	895	900	905	910	915	919	924	929	934	
890	939	944	949	954	959	963	968	973	978	983	
891	988	993	998	*002	*007	*012	*017	*022	*027	*032	
892	95 036	041	046	051	056	061	066	071	075	080	
893	085	090	095	100	105	109	114	119	124	129	
894	134	139	143	148	153	158	163	168	173	177	
895	182	187	192	197	202	207	211	216	221	226	
896	231	236	240	245	250	255	260	265	270	274	
897	279	284	289	294	299	303	308	313	318	323	
898	328	332	337	342	347	352	357	361	366	371	
899	376	381	386	390	395	400	405	410	415	419	
900	424	429	434	439	444	448	453	458	463	468	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
900	95 424	429	434	439	444	448	453	458	463	468	
901	472	477	482	487	492	497	501	506	511	516	
902	521	525	530	535	540	545	550	554	559	564	
903	569	574	578	583	588	593	598	602	607	612	
904	617	622	626	631	636	641	646	650	655	660	
905	665	670	674	679	684	689	694	698	703	708	
906	713	718	722	727	732	737	742	746	751	756	
907	761	766	770	775	780	785	789	794	799	804	
908	809	813	818	823	828	832	837	842	847	852	
909	856	861	866	871	875	880	885	890	895	899	
910	904	909	914	918	923	928	933	938	942	947	5
911	952	957	961	966	971	976	980	985	990	995	1 0.5
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	2 1.0
913	96 047	052	057	061	066	071	076	080	085	090	3 1.5
914	095	099	104	109	114	118	123	128	133	137	4 2.0
915	142	147	152	156	161	166	171	175	180	185	5 2.5
916	190	194	199	204	209	213	218	223	227	232	6 3.0
917	237	242	246	251	256	261	265	270	275	280	7 3.5
918	284	289	294	298	303	308	313	317	322	327	8 4.0
919	332	336	341	346	350	355	360	365	369	374	9 4.5
920	379	384	388	393	398	402	407	412	417	421	
921	426	431	435	440	445	450	454	459	464	468	
922	473	478	483	487	492	497	501	506	511	515	
923	520	525	530	534	539	544	548	553	558	562	
924	567	572	577	581	586	591	595	600	605	609	
925	614	619	624	628	633	638	642	647	652	656	
926	661	666	670	675	680	685	689	694	699	703	
927	708	713	717	722	727	731	736	741	745	750	
928	755	759	764	769	774	778	783	788	792	797	
929	802	806	811	816	820	825	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	
931	895	900	904	909	914	918	923	928	932	937	
932	942	946	951	956	960	965	970	974	979	984	
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	
934	97 035	039	044	049	053	058	063	067	072	077	4
935	081	086	090	095	100	104	109	114	118	123	1 0.4
936	128	132	137	142	146	151	155	160	165	169	2 0.8
937	174	179	183	188	192	197	202	206	211	216	3 1.2
938	220	225	230	234	239	243	248	253	257	262	4 1.6
939	267	271	276	280	285	290	294	299	304	308	5 2.0
940	313	317	322	327	331	336	340	345	350	354	6 2.4
941	359	364	368	373	377	382	387	391	396	400	7 2.8
942	405	410	414	419	424	428	433	437	442	447	8 3.2
943	451	456	460	465	470	474	479	483	488	493	9 3.6
944	497	502	506	511	516	520	525	529	534	539	
945	543	548	552	557	562	566	571	575	580	585	
946	589	594	598	603	607	612	617	621	626	630	
947	635	640	644	649	653	658	663	667	672	676	
948	681	685	690	695	699	704	708	713	717	722	
949	727	731	736	740	745	749	754	759	763	768	
950	772	777	782	786	791	795	800	804	809	813	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
950	97 772	777	782	786	791	795	800	804	809	813	
951	818	823	827	832	836	841	845	850	855	859	
952	864	868	873	877	882	886	891	896	900	905	
953	909	914	918	923	928	932	937	941	946	950	
954	955	959	964	968	973	978	982	987	991	996	
955	98 000	005	009	014	019	023	028	032	037	041	
956	046	050	055	059	064	068	073	078	082	087	
957	091	096	100	105	109	114	118	123	127	132	
958	137	141	146	150	155	159	164	168	173	177	
959	182	186	191	195	200	204	209	214	218	223	
960	227	232	236	241	245	250	254	259	263	268	
961	272	277	281	286	290	295	299	304	308	313	
962	318	322	327	331	336	340	345	349	354	358	5 1 0.5 2 1.0 3 1.5 4 2.0 5 2.5 6 3.0 7 3.5 8 4.0 9 4.5
963	363	367	372	376	381	385	390	394	399	403	
964	408	412	417	421	426	430	435	439	444	448	
965	453	457	462	466	471	475	480	484	489	493	
966	498	502	507	511	516	520	525	529	534	538	
967	543	547	552	556	561	565	570	574	579	583	
968	588	592	597	601	605	610	614	619	623	628	
969	632	637	641	646	650	655	659	664	668	673	
970	677	682	686	691	695	700	704	709	713	717	
971	722	726	731	735	740	744	749	753	758	762	
972	767	771	776	780	784	789	793	798	802	807	
973	811	816	820	825	829	834	838	843	847	851	
974	856	860	865	869	874	878	883	887	892	896	
975	900	905	909	914	918	923	927	932	936	941	
976	945	949	954	958	963	967	972	976	981	985	
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	
978	99 034	038	043	047	052	056	061	065	069	074	
979	078	083	087	092	096	100	105	109	114	118	
980	123	127	131	136	140	145	149	154	158	162	
981	167	171	176	180	185	189	193	198	202	207	4 1 0.4 2 0.8 3 1.2 4 1.6 5 2.0 6 2.4 7 2.8 8 3.2 9 3.6
982	211	216	220	224	229	233	238	242	247	251	
983	255	260	264	269	273	277	282	286	291	295	
984	300	304	308	313	317	322	326	330	335	339	
985	344	348	352	357	361	366	370	374	379	383	
986	388	392	396	401	405	410	414	419	423	427	
987	432	436	441	445	449	454	458	463	467	471	
988	476	480	484	489	493	498	502	506	511	515	
989	520	524	528	533	537	542	546	550	555	559	
990	564	568	572	577	581	585	590	594	599	603	
991	607	612	616	621	625	629	634	638	642	647	
992	651	656	660	664	669	673	677	682	686	691	
993	695	699	704	708	712	717	721	726	730	734	
994	739	743	747	752	756	760	765	769	774	778	
995	782	787	791	795	800	804	808	813	817	822	
996	826	830	835	839	843	848	852	856	861	865	
997	870	874	878	883	887	891	896	900	904	909	
998	913	917	922	926	930	935	939	944	948	952	
999	957	961	965	970	974	978	983	987	991	996	
1000	00 000	004	009	013	017	022	026	030	035	039	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

TABLE OF USEFUL NUMBERS

	Symbol	Numerical Value	Logarithm
Diagonal of square whose sides is 1	$\sqrt{2}$	1.4142	0.15052
Side of square whose diagonal is 1	$\frac{1}{2}\sqrt{2}$	0.70710	9.84948
Diagonal of cube whose edge is 1	$\sqrt{3}$	1.7321	0.23856
Edge of cube whose diagonal is 1	$\frac{1}{3}\sqrt{3}$	0.57735	9.76144
Circumference of circle whose diameter is 1	π	3.1416	0.49715
Diameter of circle whose circumference is 1	$\frac{1}{\pi}$	0.31831	9.50285
One meter in yards		1.0936	0.03887
One yard in meters		0.91439	9.96113
One centimeter in inches		0.39370	9.59517
One inch in centimeters		2.5400	0.40483
One square meter in square feet		10.764	1.03198
One square foot in square meters		0.092901	8.96802
One cubic centimeter in cubic inches		0.061025	8.78551
One cubic inch in cubic centimeters		16.387	1.21449
One liter in gallons		0.26418	9.42190
One gallon in liters		3.7853	0.57810
One kilogram in pounds		2.2046	0.34333
One pound in kilograms		0.45359	9.65667

LOGARITHMS

Serial 5254 Edition 1

EXPONENTS

1. **Use of Logarithms.**—For every number there is a corresponding number called a logarithm. By using logarithms, it is possible to shorten the time and labor involved in computations, as numbers can then be multiplied together or one number can be divided by another with ease, and the powers or roots of numbers can be found without much difficulty. In fact, some computations that could not be carried out conveniently by the usual arithmetical processes are rendered comparatively simple by the aid of logarithms. The use of logarithms would be much more common if they were more generally understood, and to obtain this understanding is not difficult.

Later on it will be shown how the logarithm of any number can be obtained from a table and also how logarithms are used. But, before such explanations can be made, it is necessary to explain certain fundamental principles on which logarithms are based.

2. **Powers of Numbers.**—If two or more numbers are multiplied together, each of the numbers is called a *factor* of the product. For example, 3 and 4 are factors of 12, because $3\times4=12$. Similarly, 2, 6, and 10 are factors of 120, since $2\times6\times10=120$.

When any number is multiplied by itself one or more times, the product is known as a power of the number. The power is called the *second power*, *third power*, *fourth power*, etc., according to the number of times the given number is used as a factor. Thus, 49, which equals 7×7, is the second power of 7; 125, which equals $5\times5\times5$, is the third power of 5; and 64,

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which equals $2\times2\times2\times2\times2\times2$, is the sixth power of 2. The second power of a number is commonly called the *square* of the number, and the third power is called the *cube*.

3. Use of Exponents.—For the sake of brevity, a power of a number is generally indicated by writing the number and placing at its right and near its top a small figure that shows how many times the number is to be used as a factor. Such a small figure is known as an exponent. For example, the fifth power of 3 is written 3^5 and is read either as *three to the fifth power* or as *three exponent five.* The numerical value of the power that is so indicated may be found by using the number as a factor as many times as is called for by the exponent. Thus, $3^5=3\times3\times3\times3\times3=243$. A number without a written exponent may be considered to have the exponent 1, and any number with the exponent 1 is equal to the number itself. For instance, the number 10 is equal to 10^1, and 6^1 is equal to 6.

4. Product of Two or More Powers of a Number.—If two or more indicated powers of the same number are multiplied together, the product is equal to the power of the number in which the exponent is the sum of the exponents in the factors. For example, to find the product of 2^3 and 2^5, the exponents 3 and 5 are added and the sum 8 is written as the exponent of 2 in the product; that is, $2^3\times2^5=2^{3+5}=2^8$. Similarly, $3^5\times3^4\times3=3^{5+4+1}=3^{10}$.

5. Quotient of Two Powers of a Number.—When one power of a number is divided by another power of the same number, the exponent of the number in the quotient is found by subtracting the exponent in the divisor from the exponent in the dividend. For example, if it is desired to divide 2^8 by 2^3, the exponent 3 in the divisor is subtracted from the exponent 8 in the dividend, and the difference 5 is written as the exponent of 2 in the quotient. In other words, $2^8\div2^3=2^{8-3}=2^5$. Similarly, $10^4\div10^2=10^{4-2}=10^2$.

6. Number With Zero as Exponent.—Any number with exponent 0 (zero) is equal to 1. For example, $10^0=1$. The truth of this equality can be shown by dividing a power of 10 by the same power. According to the preceding article, $10^2 \div 10^2 = 10^{2-2} = 10^0$. Also, when any number is divided by itself, the quotient is 1 and, therefore, $10^2 \div 10^2 = 1$. As the two quotients must be equal, $10^0 = 1$.

7. Positive and Negative Quantities.—In arithmetical subtraction a smaller number is always subtracted from a larger number, but in some mathematical work it is necessary to subtract a larger quantity from a smaller quantity. For instance, it may be required to divide 10^3 by 10^5, in which case the quotient would be 10^{3-5}. When a larger number is subtracted from a smaller number, the difference is a *negative number*, or a number that is less than zero. The subtraction is carried out by subtracting the smaller number from the larger number and placing a minus sign ($-$) before the result to indicate that it is negative. Thus, 5 is subtracted from 3 by subtracting 3 from 5 and writing a minus sign before the difference 2; that is, $3 - 5 = -2$. The -2 means 2 less than zero. A number that is greater than zero is called a *positive number*. Such a number may be designated by writing a plus sign ($+$) before it, but it usually is written without any sign.

The distinction between positive and negative quantities may be shown by the following simple illustrations. Temperatures above zero degrees are often considered positive, and those below zero are called negative; thus, a temperature of $42°$ or $+42°$ indicates 42 degrees above zero, and a temperature of $-15°$ indicates 15 degrees below zero. A profit in business may be considered as a positive quantity, and a loss as a negative quantity. A height above sea level may be called a positive elevation, and a depth below sea level a negative elevation. In general, the terms positive and negative may be used to distinguish between quantities that are of the same kind but are measured in opposite directions.

8. Negative Exponents.—A negative exponent is one preceded by a minus sign. Such an exponent is obtained when one power of a number is divided by a higher power of the same number. For example, when 10^3 is divided by 10^5, the quotient is $10^{3-5} = 10^{-2}$, in which there is a negative exponent.

It can be shown that a number with a negative exponent is equal to 1 divided by the number with an equal positive exponent. Thus, if the division of 10^3 by 10^5 were performed by the usual arithmetical process, the quotient would equal

$$\frac{10\times10\times10}{10\times10\times10\times10\times10}=\frac{1}{10\times10}=\frac{1}{10^2}$$

Therefore, $10^{-2}=\frac{1}{10^2}$.

A number with a negative exponent is referred to as a *negative power*. For instance, 10^{-2} is a negative power of 10; it is read *10 to the minus 2 power*.

9. Roots of Numbers.—The process of extracting a root of a number is the reverse of raising a number to a power. Thus, the square root of a number is extracted by finding the factor which when multiplied by itself produces the given number; the cube root of a number is extracted by finding one of the three equal factors of that number; and the sixth root is extracted by finding one of six equal factors.

The fact that a root of a number is to be extracted is usually indicated by means of a radical sign placed in front of the number; also, a small number, called an *index*, is written above the radical sign to show what root is desired. Usually a vinculum $\overline{}$ is connected with the radical sign and placed over the quantity to which the radical sign applies. For instance, $\sqrt[3]{8}$ indicates that the cube root of 8 is to be extracted. In the case of square root, it is not necessary to use the index 2, as the radical sign without an index is understood to indicate square root. Thus, $\sqrt{9}$ indicates the square root of 9.

It is sometimes necessary to raise a number to a power and to find the root of that power. For example, $\sqrt[3]{8^2}$ indicates that it is desired to extract the cube root of the square of 8. The value of the root may be found by computing the square of 8, which is 64, and extracting the cube root of 64.

10. Fractional Exponents.—The fact that a number is to be raised to a power and a root of that power is to be extracted may also be indicated by writing the number with a fractional exponent. The numerator of the fraction denotes the power to which the number is to be raised and the denominator shows what root is to be extracted. For instance, $\sqrt[3]{8^2}$ may be written as $8^{\frac{2}{3}}$.

Since any number may be considered to have an exponent 1, the fact that a root of a number is to be extracted may be indicated by writing the number with a fractional exponent in which the numerator is 1 and the denominator is the index of the desired root. Thus, $215^{\frac{1}{3}}$ means the same as $\sqrt[3]{215}$, or the cube root of 215. A fractional exponent may be either a common fraction or a decimal. The meaning of a decimal exponent is readily seen if it is imagined to be replaced by its equivalent common fraction in which the denominator is a multiple of 10. For example, $10^{1.81}$ is equivalent to $10^{\frac{181}{100}}$, which equals $\sqrt[100]{10^{181}}$, or the hundredth root of 10 raised to the 181st power. A number with a fractional exponent is called a *fractional power*.

The principles stated in Arts. 4 and 5 in regard to multiplication and division of powers apply whether the exponents are whole numbers or fractions, or whether they are positive or negative numbers. For example, the exponent in the product of 10^5 and 10^{-2} is equal to the sum of 5 and -2, which is 3; that is, $10^5 \times 10^{-2} = 10^{5-2} = 10^3$. The product of $10^{1.2}$ and $10^{1.3}$ is $10^{1.2+1.3} = 10^{2.5}$, the fractional exponents 1.2 and 1.3 being added as would be done for whole numbers. To illustrate a case in division in which the exponents are fractions, let it be required to divide $10^{2.1}$ by $10^{1.5}$. Since the exponent in the quotient is equal to the difference between the exponents 1.5 and 2.1, $10^{2.1} \div 10^{1.5} = 10^{2.1-1.5} = 10^{0.6}$.

EXAMPLES FOR PRACTICE

Multiply:	Divide:
(*a*) 8^2 by 8^3.	(*e*) 2^{10} by 2^2.
(*b*) 3^2 by 3^4.	(*f*) 12^5 by 12^5.
(*c*) 10^4 by 10^{-1}.	(*g*) 10^2 by 10^3.
(*d*) $10^{0.5}$ by $10^{0.45}$.	(*h*) $10^{3.1}$ by $10^{0.9}$.

Express the result in each case as a number with an exponent. Try to solve all these examples before looking for the answers in the following list.

Ans.—(*a*) 8^5; (*b*) 3^6; (*c*) 10^3: (*d*) $10^{0.95}$; (*e*) 2^8; (*f*) 12^0; (*g*) 10^{-1}; (*h*) $10^{2.2}$.

LOGARITHMS OF NUMBERS

DEFINITIONS

11. Definition of Logarithms.—Every number can be expressed as a power of 10. A number that is an integral power of 10, like 100 or 1,000, can be expressed as 10 with an exponent that is a whole number. Thus, $100=10^2$ and $1{,}000=10^3$. Also a number that is not an integral power of 10, like 2 or 34, can be expressed as a power of 10 with an exponent that is either wholly or partly decimal. For instance, 2 is very nearly equal to $10^{0.30103}$, and 34 is very nearly equal to $10^{1.53148}$.

In the system of logarithms that is commonly used, the logarithm of a number is the exponent of the power to which 10 must be raised to produce the given number. By this definition, the logarithm of 100 is 2, because the exponent of the power to which 10 must be raised to produce 100 is 2. Likewise, since $1{,}000=10^3$, the logarithm of 1,000 is 3. The number 10 is the *base* of these logarithms.

The abbreviation *log* is generally written instead of the words *the logarithm of*. For instance, the notation log 100 means the logarithm of 100. The abbreviation *log* should not be capitalized or followed by a period.

12. Logarithms That Are Whole Numbers.—Positive integral powers of 10 may be arranged as follows:

$$1=10^0 \qquad 1{,}000=10^3$$
$$10=10^1 \qquad 10{,}000=10^4$$
$$100=10^2 \qquad 100{,}000=10^5$$

Since the logarithm of a number is the exponent of the power to which 10 must be raised to produce that number, it follows that

$$\log 1 \quad =0 \qquad \log 1{,}000 \quad =3$$
$$\log 10 \quad =1 \qquad \log 10{,}000 \quad =4$$
$$\log 100=2 \qquad \log 100{,}000=5$$

Negative integral powers of 10 may also be arranged as follows:

$$0.1 = \frac{1}{10} = 10^{-1} \qquad 0.001 = \frac{1}{10^3} = 10^{-3}$$

$$0.01 = \frac{1}{10^2} = 10^{-2} \qquad 0.0001 = \frac{1}{10^4} = 10^{-4}$$

The logarithms of these powers are

$$\log 0.1 = -1 \qquad \log 0.001 = -3$$

$$\log 0.01 = -2 \qquad \log 0.0001 = -4$$

In general, the logarithm of any number that is an integral power of 10, or a number that consists of 1 and ciphers, is a whole number. If the number is greater than 1, the logarithm is positive and is equal to the number of ciphers that follow the 1. For the number 1 itself, the logarithm is zero. In case the number is less than 1, the logarithm is negative and is numerically one more than the number of ciphers between the decimal point and the 1.

13. Characteristic and Mantissa.—The logarithm of any number between 1 and 10 is zero plus a decimal, for it is more than 0 and less than 1. Likewise, the logarithm of a number between 10 and 100 is 1 plus a decimal, because it is more than 1 and less than 2; and the logarithm of any number between 100 and 1,000 is 2 plus a decimal. Also, since log 1 is 0 and log 0.1 is -1, the logarithm of any decimal number between 0.1 and 1 is -1 plus a decimal; and, since log 0.01 is -2, the logarithm of a number between 0.01 and 0.1 is -2 plus a decimal. In general, therefore, the logarithm of a number consists of two parts, namely, a whole number, which is called the *characteristic*, and a decimal, which is called the *mantissa*. For example, to produce 20, the base 10 must have an exponent of 1.30103, expressed to five decimal places, or $10^{1.30103} = 20$. Hence, $\log 20 = 1.30103$, in which 1 is the characteristic and .30103 is the mantissa. The characteristic of a logarithm may be positive or negative, depending on whether the number is greater or less than 1. *The mantissa is always positive;* it is very important to remember this fact.

The characteristic of the logarithm of any number can be readily determined by inspection of the number. However, the mantissa cannot usually be determined in this manner. If the number is an integral power of 10, as are the numbers considered in the preceding article, the mantissa is zero. For any other number, the mantissa can be calculated only by a laborious process and is generally found by means of a table of logarithms, such as that given at the beginning of this text.

DETERMINATION OF CHARACTERISTIC OF A LOGARITHM

14. Characteristics for Numbers Greater Than 1.—The term *integral figures* is used to denote either all figures of a whole number or the figures preceding the decimal point in a number that is partly decimal. Also, the term *decimal figures* is applied to the figures that follow the decimal point in a number that is either wholly or partly decimal.

The characteristic of the logarithm of any number greater than 1 is either zero or a positive number, and is one less than the number of integral figures in the given number. For example, the characteristic of the logarithm of 356 is 2, because 356 is a whole number having three figures and 2 is 1 less than 3. Likewise, the characteristic of log 31.245 is 1, because 31.245 contains two integral figures. The characteristic of log 7.53 is 0 and that of log 1,728.0036 is 3 for similar reasons.

When the characteristic of a logarithm is 0, the 0 is not always written, but it is generally considered preferable to show the 0.

15. Characteristics for Numbers Wholly Decimal.—The characteristic of the logarithm of any number less than 1 is negative and is numerically one more than the number of ciphers immediately following the decimal point in the given number. The fact that the characteristic is negative may be indicated by writing a minus sign over it. Thus, log 0.2 may be written as $\bar{1}.30103$. It would not be correct to place a minus sign in front of the characteristic, because when so written the minus sign would indicate that both the characteristic and the mantissa are negative, whereas the mantissa is always positive.

Thus, -1.30103 means $-1-.30103$, while $\bar{1}.30103$ means $-1+.30103$.

For convenience, it is customary to change a logarithm with a negative characteristic to a positive form by adding 10 to the characteristic and writing minus 10 after the mantissa of the logarithm. For example, $\bar{1}.30103$, which is the logarithm of 0.2, can be changed to a positive form by adding 10 to the negative characteristic -1 and by writing -10 after the mantissa. The logarithm is then $9.30103-10$. Since 10 is added and subtracted, the value of the logarithm is not affected and only its form is changed. In the logarithm $9.30103-10$, the characteristic has two parts: the positive part 9 that precedes the mantissa, and the negative part -10 that follows the mantissa.

The positive part of the characteristic of the logarithm of a number that is wholly decimal may be obtained directly by subtracting from 9 a number equal to the number of ciphers immediately following the decimal point in the given number. Thus, the positive part of the characteristic of log 0.067 is $9-1=8$, because there is one cipher immediately to the right of the decimal point; and the positive part of the characteristic of log 0.0005 is 6, since $9-3=6$. In case no cipher follows the decimal point, the characteristic is 9. For example, the positive part of the characteristic of log 0.5007 is $9-0=9$, there being no cipher immediately after the decimal point.

It is well for the beginner to write -10 after the mantissa of each logarithm in which the characteristic has been increased by 10 for the purpose of changing the logarithm to a positive form. The experienced computer usually omits the -10 but always understands it to be a part of the characteristic.

16. Rules for Finding Characteristics.—The characteristic of the logarithm of any number may be found by one of the following two rules:

Rule I.—*For a number greater than 1, the characteristic is one less than the number of integral figures in the number.*

Rule II.—*For a number that is wholly decimal, the positive part of the characteristic is equal to the difference between 9 and* *the number of ciphers immediately following the decimal point in the number.*

It should be kept in mind that the characteristic of the logarithm of any number is determined entirely by the position of the decimal point in the number.

EXAMPLES FOR PRACTICE

What are the characteristics of the logarithms of the following numbers? Use the positive form of the characteristic in each case.

		Ans.
(*a*)	219	(*a*) 2
(*b*)	0.0125	(*b*) $8-10$
(*c*)	3,007	(*c*) 3
(*d*)	0.00000375	(*d*) $4-10$
(*e*)	4.245	(*e*) 0
(*f*)	0.00542	(*f*) $7-10$
(*g*)	0.000000795	(*g*) $3-10$
(*h*)	1.05	(*h*) 0
(*i*)	0.1497	(*i*) $9-10$
(*j*)	700	(*j*) 2
(*k*)	27.4	(*k*) 1
(*l*)	0.0001	(*l*) $6-10$

DETERMINATION OF MANTISSA OF A LOGARITHM

MANTISSAS FOR NUMBERS WITH NOT MORE THAN FOUR SIGNIFICANT FIGURES

17. Table of Mantissas for Whole Numbers From 1 to 100.—The labor involved in computing directly even a single mantissa is so great that the use of logarithms is impracticable without a table. Mantissas are generally unending decimals. Therefore, when mantissas are computed for a table, the results can be carried to as many figures as are desired. The mantissas given in the Table of Common Logarithms at the beginning of this text were computed so as to be correct to five figures. By means of this table, the mantissa of the logarithm of any number can be readily determined to five figures.

The mantissas of the logarithms of whole numbers from 1 to 100 are given on page 1 of the table. In the column headed N. are the numbers; to the right of each number, in the adjoining column headed Log., is the mantissa of the corresponding logarithm. Although tables like that given in this text are called tables of logarithms, they are actually tables of mantissas, since the characteristics are not printed. Furthermore, the mantissas given in such tables are not preceded by decimal points, because in determining mantissas by means of a table it is convenient to treat them as whole numbers. However, it should be remembered that a mantissa is entirely decimal and that in every logarithm the mantissa should be preceded by a decimal point and a characteristic. Thus, the mantissa of log 2 is given as 30103, but the full value of the logarithm with the characteristic and the decimal point is 0.30103. Also, the mantissa of log 30 is given as 47712, but the complete logarithm is 1.47712.

The part of the table on page 1 may occasionally be found useful, especially by the beginner; but many tables of logarithms are printed without this part, because the mantissas given on pages 2 to 19 can be used for finding the logarithms of numbers below 100. In general, pages 2 to 19 should be consulted when logarithms are being used.

18. Significant Figures.—The logarithms of the numbers 3, 30, 300, 0.3, 0.03, and 0.003 all have the same mantissa 47712, but different characteristics. Thus, $\log 3 = 0.47712$, $\log 30 = 1.47712$, $\log 300 = 2.47712$, $\log 0.3 = 9.47712 - 10$, $\log 0.03 = 8.47712 - 10$, and $\log 0.003 = 7.47712 - 10$. Hence, in determining the mantissas of logarithms, it is not necessary to consider ciphers at the end of any number or at the beginning of a number that is wholly decimal. The figures of a number that remain after such ciphers have been dropped are the significant figures. Thus, each of the numbers just given has only one significant figure, namely, 3. Similarly, each of the numbers 256, 25.6, 25,600, and 0.0256 has the three significant figures 256. In the case of the number 1,072, all four given figures are significant figures, because a cipher must be considered as a significant figure when it occurs between two other significant figures. The number 107,200 likewise has the four significant figures 1072, as neither of the ciphers following the 2 is a significant figure. It is important to remember that *all* *numbers with the same significant figures have the same mantissa in their logarithms.*

EXAMPLES FOR PRACTICE

Write the significant figures of each of the following numbers:

(*a*) 2,348.
(*b*) 23.48.
(*c*) 400.
(*d*) 400.3.
(*e*) 0.075.
(*f*) 0.0705.
(*g*) 91,630.
(*h*) 91,600.

Ans.—(*a*) 2348; (*b*) 2348; (*c*) 4; (*d*) 4003; (*e*) 75; (*f*) 705; (*g*) 9163; (*h*) 916

19. Mantissas of Logarithms of Whole Numbers With Four Figures.—On pages 2 to 19 of the table are given the mantissas of the logarithms of all whole numbers from 1,000 to 10,000. In the columns headed N. are numbers running from 100 to 999. One of these numbers corresponds to the first three figures of the given number. To the right are ten columns headed 0, 1 2, and so on to 9. One of these numbers is the fourth figure of the given number. The required mantissas are found from the figures in the ten columns that are headed 0 to 9.

In the column headed 0, either three or five figures are printed. Where the first two figures would be the same as those in the line above, they are not repeated, and a blank space is left. In the columns headed 1, 2, 3, etc., only the last three figures of the mantissas are given. The first two figures are the same as those of the first mantissa on the same line, except where a star (*) occurs. The star indicates that the first two figures of the mantissa are the same as those in the first mantissa in the line below.

To find from the table the mantissa of the logarithm of a whole number with four figures, look in the column headed N. and find the first three figures of the given number. Then follow along the line across the page until the column is reached which is headed with the fourth figure of the given number. In this line and in this column will be found the last three figures of the mantissa. For the first two figures, take the first two figures of the mantissa in the same line, in the column headed 0, unless the printed mantissa is preceded by a star. In that case, take the corresponding two figures in the line below.

Example 1.—Find the logarithm of 1,040.

Solution.—The first step is to look in the column headed N. of the table for 104, the first three figures of the given number. This is found on page 2. In the same line with 104 and in the column headed 0, which is the fourth figure of the given number, are found the figures 703. These are the last three figures of the desired mantissa. As there is a blank space to the left of these figures, the first two figures 01 of the mantissa are taken from the first line above that contains five figures. Therefore, the mantissa of log 1,040 is 01703. Since there are four integral figures in 1,040, the characteristic of log 1,040 is 3, or 1 less than 4, and log 1,040 is 3.01703. Ans.

Example 2.—Find the logarithm of 1,047.

Solution.—The figures 104 are located in the column headed N., as in the preceding example. Then, in the line with these three figures and in the column headed 7, which is the fourth figure of the given number, the figures 995 are found. These are the last three figures of the required mantissa. On the same line in the column headed 0, only three figures are printed. Hence, the first two figures of the mantissa are taken from the line above and the mantissa is 01995. Since the characteristic of log 1,047 is 3, the required logarithm is 3.01995. Ans.

Example 3.—Find log 1,048.

Solution.—First, the figures 104 are located as before in the column headed N. Then, in the same line and in the column headed 8, are found the last three figures 036 of the required mantissa. However, as these figures are preceded by a star, the first two figures of the mantissa are the same as those on the line below in the column headed 0, and are 02. Thus, the required mantissa is 02036, and log 1,048 = 3.02036. Ans.

EXAMPLES FOR PRACTICE

Find the logarithm of each of the following numbers:

(*a*) 4,080.
(*b*) 9,700.
(*c*) 3,673.
(*d*) 7,854.
(*e*) 1,906.
(*f*) 4,678.

Ans.—(*a*) 3.61066; (*b*) 3.98677; (*c*) 3.56502; (*d*) 3.89509; (*e*) 3.28012; (*f*) 3.67006

20. Mantissa for Any Number With Not More Than Four Significant Figures.—As previously explained, all numbers that have the same significant figures have the same mantissa in their logarithms. Therefore, in finding the mantissa of the logarithm of a whole number that consists of four significant figures followed by ciphers, these ciphers may be disregarded. Also, if a number with four significant figures is wholly or partly decimal, the decimal point and any ciphers preceding the significant figures may be disregarded and the number may be treated as a whole number with the same four significant figures. For example, the mantissa of log 104,800, log 10.48, or log 0.01048 is found as if the number were 1,048. The characteristics of the logarithms of these numbers would, of course, be different and would be determined by the rules in Art. 16.

EXAMPLE 1.—Find log 25.47.

SOLUTION.—The mantissa of log 25.47 is the same as that of log 2,547, which is found as follows: The first three figures 254 are located in the table in the column headed N. In line with these figures and in the column headed 7 are found the figures 603, which are the last three figures of the mantissa. The next step is to go back to the column headed 0 and to glance upward until the first two figures 40 of the mantissa are reached. Thus, the required mantissa is found to be 40603. Since the characteristic of log 25.47 is 1, log $25.47 = 1.40603$. Ans.

EXAMPLE 2.—What is the logarithm of 107,300?

SOLUTION.—To find the mantissa of log 107,300, disregard the last two ciphers and consider the number 1,073. Therefore, look for 107 in the column headed N., follow the line thus located to the column headed 3, and record the figures 060 found there. Since these figures are preceded by a star, find the first two figures of the mantissa, namely, 03, in the column headed 0 and in the line below. The mantissa is, therefore, 03060. In this case, the characteristic is 5. Hence, log $107{,}300 = 5.03060$. Ans.

EXAMPLE 3.—Find log 0.06313.

SOLUTION.—To find the mantissa of this logarithm, treat the number as if it were 6,313. Thus, locate 631 in the column headed N.; find the last three figures of the mantissa, or 024, in the same line in the column headed 3; and in this case find the first two figures of the mantissa, or 80, in the same line in the column headed 0. The mantissa is 80024. The positive part of the characteristic of the logarithm of the given number is $9-1$, or 8, and log $0.06313 = 8.80024-10$. Ans.

21. If the given number contains less than four significant figures, the significant figures are written and a cipher or ciphers are annexed to make up a number of four figures. The mantissa of this number is the required mantissa. For example, if the given number is 104, one cipher is annexed and the number used in finding the mantissa is 1,040. Similarly, if the given number is 0.368, one cipher is annexed to the significant figures 368, and the number obtained is 3,680.

In the case of the number 0.0036, two ciphers are annexed to the significant figures 36 and the number 3,600 is obtained. However, the mantissa of the logarithm of any number containing only one or two significant figures may be taken directly from page 1 of the table. It is the mantissa of the whole number consisting of the same one or two significant figures. Thus, the mantissa of log 0.0036 is the same as the mantissa of log 36, which is 55630. But, as previously stated, a page like this is not always given in tables of logarithms, because the mantissas of the logarithms of numbers with one or two significant figures can be easily found from pages 2 to 19.

EXAMPLE 1.—Find the logarithm of 476.

SOLUTION.—First a cipher is added to the given number to obtain the number 4,760, which contains four figures. Then, the first three figures 476 are located in the column headed N., and in line with these figures in the column headed 0 are found the figures 761, which are the last three figures of the required mantissa. Since there is a blank space in front of these figures in the table, the first two figures are found by glancing upward in the column headed 0 until the figures 67 are reached. The mantissa of log 476 is 67761. As the characteristic of log 476 is 2, log 476 = 2.67761. Ans.

EXAMPLE 2.—Find the logarithm of 0.0026, (*a*) by using page 1 of the table, and (*b*) without using page 1.

SOLUTION.—(*a*) The significant figures of 0.0026 are 26, and the mantissa given on page 1 for log 26 is 41497. Therefore, log 0.0026 = 7.41497 − 10. Ans.

(*b*) If page 1 is not available, the mantissa of log 0.0026 is determined by first writing the significant figures 26 and annexing two ciphers to obtain the number 2,600, and then finding the mantissa of log 2,600 in the following manner: Opposite the figures 260 in the column headed N. on page 5 and in the column headed 0 are the three figures 497. Since the first two figures of the mantissa are 41, the mantissa is 41497. Therefore, log 0.0026 = 7.41497 − 10. Ans.

EXAMPLES FOR PRACTICE

Find the logarithms of the following numbers:

(*a*) 0.062.	(*e*) 89.42.
(*b*) 620.	(*f*) 10,000.
(*c*) 21.4.	(*g*) 35,030.
(*d*) 0.000067.	(*h*) 575,900.

Ans.—(*a*) 8.79239 − 10; (*b*) 2.79239; (*c*) 1.33041; (*d*) 5.82607 − 10; (*e*) 1.95143; (*f*) 4.00000; (*g*) 4.54444; (*h*) 5.76035

MANTISSAS FOR NUMBERS WITH MORE THAN FOUR SIGNIFICANT FIGURES

22. General Principle of Interpolation.—The table on pages 2 to 19 can also be used for finding the mantissas of the logarithms of numbers containing more than four significant figures. Such a mantissa is intermediate between two values given in the table, and is found by a process known as *interpolation*.

The general principle involved in interpolation may be illustrated by finding the mantissa of log 31,416. Obviously, this mantissa must be greater than that of log 31,410, which is the same as for log 3,141 and is 49707, and less than that of log 31,420, which is the same as for log 3,142 and is 49721. The required mantissa may therefore be found by adding a certain number to 49707. The difference between the mantissas of log 3,142 and log 3,141 is $49721-49707=14$. Hence, for a difference of 1 unit in the fourth figure of the number there is in this case a difference of 14 units in the mantissa. Since 1 unit in the fourth place is equal to 10 units in the fifth place, an increase of 1 unit in the fifth place in the number corresponds to an increase in the mantissa of $\frac{1}{10}$ of 14, or 1.4 units. For an increase of 6 units in the fifth place, there would be an increase of $6\times1.4=8.4$ in the mantissa. As 8 is the whole number nearest to 8.4, the number to be added to the mantissa of log 3,141 is 8. The mantissa of log 31,416 is $49707+8=49715$, and log $31{,}416=4.49715$.

When a five-place table of mantissas is used for finding the logarithm of a number containing six or more significant figures, it is not practical to consider more than five figures, because the accuracy of the final result will not be increased by the use of a greater number of figures. Therefore, all significant figures beyond the fifth may be replaced by zeros and, if the sixth figure is 5 or more, the fifth figure should be increased by 1. Thus, if the given number is 31,415,426, the mantissa of log 31,415,000 or of log 31,415 should be found; and, if the given number is 31,415,926, the mantissa of log 31,416,000 or of log 31,416 should be used.

If the given number is partly or entirely decimal, it is not necessary to replace by zeros decimal figures beyond the fifth significant figure, and these figures are merely dropped; the fifth figure is increased by 1 if the sixth figure is 5 or more. For instance, if the number is 0.0700256, the mantissa would be found for 0.070026; 3.14159 would be considered 3.1416; and 279,349.3 would be considered 279,350.

23. When interpolation is necessary, the number that is to be added to a mantissa given in the table is called the *proportional part*. The difference between two consecutive mantissas in the table is called the *table difference* or the *tabular difference*. To find the mantissa for a number of five significant figures, the first step is to find the mantissa for the number that consists of only the first four significant figures of the given number. The second step is to determine the table difference for this mantissa, or the difference between this mantissa and the next larger mantissa in the table. The third step is to compute the required proportional part by multiplying one-tenth of the table difference by the fifth significant figure of the given number. The fourth step is to obtain the required mantissa by adding this proportional part to the mantissa for the number of only four significant figures. The proportional part that is used for the interpolation is the whole number nearest to the computed proportional part. The computations for determining the mantissa of log 31,416 may be conveniently arranged as follows:

mantissa of log 3,142 = 49721
mantissa of log 3,141 = 49707

table difference = 14

proportional part $=6\times 1.4=\quad 8$
mantissa of log 3,141 $=49707$

mantissa of log 31,416 $=49715$

24. Tables of Proportional Parts.—Instead of computing the proportional parts, as was done in the preceding article, the proportional parts for each table difference can be taken directly from the auxiliary tables in the columns headed P.P. (proportional parts) on pages 2 to 19. For convenient reference, the table of proportional parts for a table difference of 14 is reproduced here. In this table, the heading **14** denotes the table difference; the figures in the left-hand column represent the fifth figures of numbers with more than four significant figures; and the numbers under the heading in the right-hand column are the proportional parts for the respective fifth figures and the particular table difference. For example, in the case of the mantissa of log 31,416, the table difference, or the difference between the mantissa 49707 of log 3,141 and the mantissa 49721 of log 3,142, is 14; and the fifth figure of the given number is 6. In the table of proportional parts headed 14, the proportional part opposite 6 is found to be 8.4, or 8. As in the preceding article, this proportional part should be added to the mantissa 49707 of log 3,141 to obtain the mantissa 49715 of log 31,416.

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

25. The procedure for determining the mantissa of the logarithm of a number containing five significant figures by using the table of proportional parts is as follows:

First find from the columns of mantissas the mantissa corresponding to the first four significant figures of the given number and also the next larger mantissa in the table.

Second, compute the table difference by subtracting the first of these mantissas from the second one, and locate the auxiliary table of proportional parts for that table difference in a column headed P. P.; the desired auxiliary tablz will generally be on the same page as the mantissas.

Third, find the required proportional part in that auxiliary table opposite the fifth figure of the given number.

Fourth, add this proportional part to the mantissa corresponding to the first four figures of the given number.

Example 1.—Find the logarithm of 380.93.

Solution.—The first four significant figures of the given number are 3809, and from page 7 of the table it is found that the mantissa of log 3,809 is 58081 and that the mantissa of log 3,810 is 58092. Hence, the table difference is 58092 − 58081 = 11. The next step is to locate the auxiliary table headed 11 in the column headed P. P. It is found there that the proportional part corresponding to the fifth figure 3 of the given number is 3.3, or 3. The required mantissa is, therefore, 58081 + 3 = 58084. Since the number has three integral figures, the characteristic is 2, and log 380.93 = 2.58084. Ans.

If the two mantissas that are found directly from the table are copied for the purpose of determining the table difference, it will be convenient first to record the mantissa for the first four significant figures of the given number and then to write *above* it the mantissa for the next larger number. The complete work for finding the logarithm may be tabulated in the following manner:

mantissa of log 3,810 = 58092
mantissa of log 3,809 = 58081

table difference = 11
P.P. for 3 = 3
mantissa of log 3,809 = 58081

mantissa of log 38,093 = 58084

Thus, log 380.93 = 2.58084. Ans.

Example 2.—Find the logarithm of 1,296,728.

Solution.—For finding the mantissa, the number considered is 12,967. This mantissa is determined as follows:

mantissa of log 1,297 = 11294
mantissa of log 1,296 = 11261

table difference = 33
P.P. for 7 = 23
mantissa of log 1,296 = 11261

mantissa of log 12,967 = 11284

The characteristic is 6, and log 1,296,728 = 6.11284. Ans.

Example 3.—Find log 0.096725.

Solution —The mantissa of log 96,725 is found as follows:

mantissa of log 9,673 = 98556
mantissa of log 9,672 = 98552

table difference = 4
P.P. for 5 = 2
mantissa of log 9,672 = 98552

mantissa of log 96,725 = 98554

In this case, the positive part of the characteristic is 8. Therefore, log 0.096725 = 8.98554 − 10. Ans.

Example 4.—What is log 89.1269?

Solution.—The mantissa is the same as that of log 89,127, which is found in the following manner:

mantissa of log 8,913 = 95002
mantissa of log 8,912 = 94998

table difference = 4
P.P. for 7 = 3
mantissa of log 8,912 = 94998

mantissa of log 89,127 = 95001

The characteristic is 1 and log 89.1269 = 1.95001. Ans.

26. The calculations for obtaining the mantissa of a logarithm by interpolation can be made mentally without much difficulty. As the table differences are small numbers, the largest table difference being 44, the difference between two consecutive mantissas in the table can be found without copying the mantissas. The proportional part corresponding to the table difference and the fifth figure of the given number can then be taken directly from the proper table of proportional parts, and this proportional part can be added mentally to the mantissa that corresponds to the first four figures of the given number.

EXAMPLES FOR PRACTICE

Find the logarithms of the following numbers:

(*a*) 89.426.
(*b*) 89.428.
(*c*) 0.0010823.
(*d*) 1,923.208.
(*e*) 3.00026.
(*f*) 0.785398.
(*g*) 0.025741.
(*h*) 510.57.

Ans.—(*a*) 1.95146; (*b*) 1.95147; (*c*) 7.03435−10; (*d*) 3.28403; (*e*) 0.47717; (*f*) 9.89509−10; (*g*) 8.41063−10; (*h*) 2.70806

DETERMINATION OF ANTILOGARITHMS

FINDING ANTILOGARITHM DIRECTLY FROM TABLE

27. General Principle.—After one knows how to obtain the logarithm of a number from a table, it is not a difficult matter for him to learn how to find the number corresponding to a given logarithm. The figures that form the number corresponding to a logarithm depend entirely on the mantissa of the logarithm. The characteristic of the logarithm serves to indicate the position of the decimal point in the number. The number that corresponds to a logarithm is known as the *antilogarithm* of the given logarithm. The abbreviation *antilog* is usually written instead of the words *the antilogarithm of*.

28. Rules for Locating Decimal Point.—The rules for locating the decimal point in an antilogarithm, which can be derived by reversing the rules given in Art. 16, may be stated as follows:

Rule I.—*If the characteristic of the logarithm is positive, the number of integral figures in the antilogarithm is one more than the characteristic.*

Rule II.—*If the logarithm has a negative characteristic expressed in positive form, the antilogarithm is wholly decimal and the number of ciphers immediately following the decimal point in the antilogarithm is found by subtracting the positive part of the characteristic from 9.*

For example, if the characteristic of a logarithm is 2, the antilogarithm has 2+1, or 3, integral figures. If the characteristic is 0, there is 0+1, or 1, figure before the decimal point in the antilogarithm. Again, if the positive part of a negative characteristic that is expressed in positive form is 7, the corresponding number is wholly decimal and contains $9-7$, or 2, ciphers immediately after the decimal point. If the positive part of the negative characteristic is 9, the corresponding number in this case also is wholly decimal but there is no cipher immediately following the decimal point, since $9-9=0$.

29. Types of Problems in Finding Antilogarithms.—The mantissa of the given logarithm may or may not be found in the table. If the mantissa is found in the table, the required number has not more than four significant figures, and these are given directly. The first three are in the column headed N. on the same line with the mantissa, and the fourth is at the head of the column in which the last three figures of the mantissa are located. When the given mantissa is not found in the table, the first four significant figures of the required number correspond to the next smaller tabulated mantissa, and the fifth figure of the number may be found by interpolation. For certain computations that come up in practice, it may be sufficient to find only four figures in the number. The number corresponding to the nearest tabulated mantissa may then be taken. In the problems in this text, interpolation should be employed to find the fifth figure of the number. When five-place tables of mantissas are used, it is not practical to find more than five figures of a number from those tables.

30. Procedure in Finding Antilogarithm When Mantissa Is Given in Table.—The number corresponding to a logarithm whose mantissa is found in the table may be determined in the following manner: First, locate the first two figures of the given mantissa in the column of the table headed 0. Next glance through the mantissas beginning with these two figures until the given mantissa is found. Among these mantissas should be included also those whose last three figures are preceded by a star in the line above the one in which the first two figures are located. The first three significant figures of the required number are those in the column headed N. on the line with the last three figures of the mantissa, and the fourth significant figure of the number is that at the head of the column containing the last three figures of the mantissa. After the significant figures of the required number are determined from the mantissa of the logarithm, the decimal point is located in the number by considering the characteristic of the logarithm. If the characteristic is greater than 3, ciphers should be annexed to the four significant figures to provide the required number of integral figures.

Example 1.—Find the number whose logarithm is 3.56867.

Solution.—The first step is to locate the first two figures of the mantissa, or 56, in one of the columns headed 0. These figures are located in the column headed 0 that is on page 7 of the table. The next step is to look through the mantissas that begin with these two figures until the last three figures of the mantissa, or 867, are located. These are found in line with the number 370 of the column headed N. and in the column headed 4. Hence, the first three figures of the required number are 370 and the fourth figure is 4; that is, the significant figures in the number are 3704. Since the characteristic of the given logarithm is 3, the number should contain four integral figures, and the required number is 3,704. Ans.

Example 2.—Find the number whose logarithm is $8.40858-10$.

Solution.—In this case, the first two figures 40 of the mantissa are found in the column headed 0 on page 5 of the table, and the last three figures of the mantissa, or 858, are found in line with the number 256 of the column headed N. and in the column headed 2. The four significant figures of the required number are therefore 2562. As the positive part of the characteristic of the logarithm is 8, the number of ciphers following the decimal point in the antilogarithm is $9-8=1$. Hence, the required number is 0.02562. Ans.

Example 3.—What is the number whose logarithm is 5.81478?

Solution.—The first two figures of the mantissa, or 81, are found on page 13 of the table, and the last three figures, or 478, are located in line with the number 652 of the column headed N. and in the column headed 8. Therefore, the significant figures in the required number are 6528. Since the number must have $5+1=6$ integral figures, it is 652,800. Ans.

Example 4.—Find the number whose logarithm is 0.88081.

Solution.—The given mantissa is found on page 15 of the table in line with the number 760 of the column headed N. and in the column headed 0. Therefore, the first three figures of the number that corresponds to the mantissa are 760 and the fourth figure is 0. Since the characteristic of the given logarithm is 0, there should be one integral figure in the corresponding number, which is 7.600, or 7.6. Ans.

Example 5.—Find the number whose logarithm is 1.04727.

Solution.—The first two figures of the mantissa, or 04, are found in the column headed 0 on page 2 of the table, and the last three figures of the given mantissa, or 727, are located in line with the number 111 of the column headed N. and in the column headed 5. Hence, the significant figures in the required number are 1115, and the number is 11.15. Ans.

Example 6.—Find the number whose logarithm is 2.27091.

Solution.—The first two figures of the mantissa, or 27, are found in the column headed 0 on page 3 of the table. The last three figures 091 of the given mantissa are located in the line above in the column headed 6 and opposite the number 186 of the column headed N. The figures 091 are preceded by a star to show that the complete mantissa is 27091, not 26091. Therefore, the significant figures are 1866, and the required number is 186.6. Ans.

EXAMPLES FOR PRACTICE

Find the numbers corresponding to the following logarithms:

(*a*) 0.74453.
(*b*) 4.38471.
(*c*) 9.84510−10.
(*d*) 1.84510.
(*e*) 6.96009−10.
(*f*) 2.30428.
(*g*) 0.42439.
(*h*) 8.82458−10.

Ans.—(*a*) 5.553; (*b*) 24,250; (*c*) 0.7; (*d*) 70; (*e*) 0.0009122; (*f*) 201.5, (*g*) 2.657; (*h*) 0.06677

FINDING ANTILOGARITHM BY INTERPOLATION

31. When the mantissa of the given logarithm is not tabulated and the corresponding number is to be found correct to five significant figures, the method of procedure in determining the significant figures is as follows:

First, find the tabulated mantissa that is nearest to, but smaller than, the given mantissa, and record the four figures of the number that corresponds to this next smaller mantissa; these are the first four significant figures of the required number.

Second, obtain the table difference by subtracting the next smaller printed mantissa from the next greater one, and also compute the proportional part for the given mantissa by subtracting the next smaller printed mantissa from the given mantissa.

Third, find the fifth significant figure of the required number from the auxiliary table of proportional parts that is headed by the table difference.

Fourth, consider the characteristic and locate the decimal point in the required number or annex ciphers if the characteristic is positive and greater than 4.

The fifth figure is found in the first column of the auxiliary table, opposite either the computed proportional part or the printed proportional part that is nearest to the computed value. If the computed proportional part is midway between two proportional parts given in the table, it is customary to take for the fifth significant figure the number in the first column opposite the larger of the two proportional parts in the table.

EXAMPLE 1.—Find the number whose logarithm is 2.56871.

SOLUTION.—Since the given mantissa 56871 is not found in the table, the first step is to find the first four significant figures of the required number by determining the four figures that correspond to the next smaller printed mantissa, which is 56867. These four figures are 3704. The next step is to find the table difference, or the difference between the next smaller printed mantissa 56867 and the next greater mantissa 56879, which is 12; and also the proportional part, or the difference between the next smaller mantissa 56867 and the given mantissa 56871, which is 4. The third step is to locate the auxiliary table of proportional parts that is headed 12 and to find in the second column of that table the printed proportional part nearest to 4. This nearest proportional part is 3.6, and the number in the first column opposite it is 3. Hence, the fifth significant figure of the required number is 3, and the five significant figures are 37043.

As the characteristic of the given logarithm is 2, the required number will contain 2+1, or 3, integral figures. Therefore, the antilogarithm of 2.56871 is 370.43. Ans.

The computations may be indicated in the following manner:

Next lower mantissa = 56867 = mantissa of log 3,704

next greater mantissa = 56879
next lower mantissa = 56867
table difference = 12
fifth significant figure = 3

given mantissa = 56871
next lower mantissa = 56867
proportional part = 4

Since the characteristic is 2, the required number is 370.43. Ans.

EXAMPLE 2.—Find the number whose logarithm is 0.93036.

SOLUTION.—The given mantissa 93036 is not found in the table and the next smaller printed mantissa 93034 is therefore located. The figures in the corresponding number, which are also the first four significant figures of the required number, are 8518.

The table difference is 93039 − 93034 = 5, and the proportional part is 93036 − 93034 = 2. In the auxiliary table of proportional parts that is headed 5, the proportional part 2 is found opposite the number 4. Hence, the fifth significant figure of the required number is 4 and the five significant figures are 85184. Since the characteristic of the given logarithm is 0, there is one integral figure in the required number, which is 8.5184. Ans.

Example 3.—What is the number whose logarithm is 8.05753 − 10?

Solution.—The next smaller printed mantissa, or 05729, is found in the table in the column headed 1 and on the line with the figures 114 in the column headed N. Therefore, the first four significant figures of the required number are 1141. The table difference is 05767 − 05729 = 38 and the proportional part is 05753 − 05729 = 24. In the auxiliary table of proportional parts that is headed 38, the proportional part nearest to 24 is 22.8. The number in the first column opposite 22.8 is 6, which is the fifth significant figure of the required number, and the five significant figures of the number are 11416.

The −10 in the logarithm shows that the required number is wholly decimal, and the positive part 8 of the characteristic shows that one cipher should follow the decimal point. Hence, the number whose logarithm is 8.05753 − 10 is 0.011416. Ans.

32. The fifth significant figure of a number corresponding to a logarithm can be determined without a table of proportional parts by dividing the computed proportional part by one-tenth of the table difference. Thus, in example 3, Art. 31, where the table difference is 38 and the proportional part is 24, the fifth figure of the required number may be found as follows:

$$\frac{24}{3.8} = 6.3, \text{ or } 6.$$

EXAMPLES FOR PRACTICE

Find the numbers corresponding to the following logarithms:

(a) 3.78942.
(b) 0.50210.
(c) 8.63491 − 10.
(d) 9.07619 − 10.
(e) 7.23417.
(f) 1.44969.
(g) 0.50000.
(h) 2.98273.

Ans.—(a) 6,157.7; (b) 3.1776; (c) 0.043143; (d) 0.11918; (e) 17,146,000; (f) 28.164; (g) 3.1623; (h) 961.02

PRACTICAL APPLICATIONS OF LOGARITHMS

MULTIPLICATION BY USE OF LOGARITHMS

33. General Rule for Multiplication.—When two or more indicated powers of a number are multiplied together, the exponent of the number in the product is equal to the sum of the exponents in the factors. Since the logarithms of numbers are the exponents of the powers to which 10 must be raised to produce those numbers, the logarithm of the product of two or more numbers is equal to the sum of the logarithms of the numbers. As an illustration, let it be required to find the product of 2 and 3. The logarithm of 2 is 0.30103, the logarithm of 3 is 0.47712, and the sum of these logarithms is 0.77815, which is the logarithm of 6.

The rule for multiplication by the use of logarithms is as follows:

Rule.—*To multiply two or more numbers by using logarithms, add the logarithms of the given numbers and thus obtain the logarithm of the product. Then find the number corresponding to that logarithm.*

34. Multiplication of Numbers Greater Than 1.—When all the factors are greater than 1, the logarithm of their product is found by adding the logarithms of the factors by the usual arithmetical process.

EXAMPLE.—Find the product of 4.38, 5.217, and 83.

SOLUTION.—

$$\begin{aligned} \log 4.38 &= 0.64147 \\ \log 5.217 &= 0.71742 \\ \log 83 &= 1.91908 \\ \hline \end{aligned}$$

By addition, $\log \text{product} = 3.27797$

The number whose logarithm is 3.27797 is 1,896.6. Hence,

$$4.38 \times 5.217 \times 83 = 1{,}896.6. \quad \text{Ans.}$$

By actual multiplication, the product is found to be 1,896.58818. Therefore, in this case, the result obtained by using logarithms is correct to five figures. However, the fifth figure may not always be exact.

35. Multiplication Involving Numbers Wholly Decimal. If numbers that are wholly decimal are to be multiplied, the logarithms will contain −10's. The sum of the −10's must be included as part of the sum of the logarithms, which is the logarithm of the product. Whenever the negative part of the sum is smaller than the positive part, the indicated subtraction is actually performed in order to have the logarithm in its simplest form. If, however, the negative part of the logarithm is the larger and is more than 10, it is reduced to 10 by subtracting the necessary number from it and also from the positive part of the logarithm.

EXAMPLE 1.—Determine the product of 49.82, 0.00243, 17, and 0.97.

SOLUTION.—

$$\begin{array}{ll} \log 49.82 & = 1.69740 \\ \log 0.00243 & = 7.38561-10 \\ \log 17 & = 1.23045 \\ \log 0.97 & = 9.98677-10 \\ \hline \end{array}$$

By addition,

$$\begin{array}{ll} \log \text{product} & = 20.30023-20 \\ & = 0.30023 \end{array}$$

The number corresponding to this logarithm is 1.9963, or

$$49.82 \times 0.00243 \times 17 \times 0.97 = 1.9963. \quad \text{Ans.}$$

In this case, the positive part of the logarithm of the product is greater than the negative part, and this logarithm is reduced to the simple form 0.30023 by performing the indicated subtraction.

EXAMPLE 2.—Find the product of 49.82, 0.00243, 0.17, and 0.97.

SOLUTION.—

$$\begin{array}{ll} \log 49.82 & = 1.69740 \\ \log 0.00243 & = 7.38561-10 \\ \log 0.17 & = 9.23045-10 \\ \log 0.97 & = 9.98677-10 \\ \hline \end{array}$$

By addition,

$$\begin{array}{ll} \log \text{product} & = 28.30023-30 \\ & \ 20 \qquad\quad -20 \\ \hline & \ 8.30023-10 \end{array}$$

The corresponding number is 0.019963. Ans.

In this case, the characteristic of the positive part of the logarithm of the product is 28 and the negative part is 30. Therefore, the logarithm should be simplified by reducing the negative part to 10, and this is accomplished by subtracting 20 from both the negative part and the positive part of the logarithm. The logarithm of the required product then becomes 8.30023 − 10.

EXAMPLE 3.—Find the product of 2,578.6 and 0.04326.

SOLUTION.—

$$\begin{aligned} \log 2{,}578.6 &= 3.41138 \\ \log 0.04326 &= 8.63609-10 \\ \hline \log \text{product} &= 12.04747-10 \\ &= 2.04747 = \log 111.55 \end{aligned}$$

Thus, 2,578.6 × 0.04326 = 111.55. Ans.

36. Care in Use of Abbreviation *log*.—In setting down logarithmic calculations, care must be taken to show properly which numbers are logarithms and which are numbers corresponding to logarithms. Since *log* is the abbreviation for *the logarithm of*, it is to be written only before the number whose logarithm either is known or is to be found. It is never written before the logarithm. This point should be carefully noted. In example 3 of Art. 35, it is correct to write 2.04747 = log 111.55; but it would not be correct to write log 2.04747 = 111.55, since 111.55 is not the logarithm of 2.04747.

EXAMPLES FOR PRACTICE

Find the products of the following by the use of logarithms:

(*a*) 100 × 32 × 31.64.
(*b*) 23.1 × 59.64 × 7.863.
(*c*) 0.00354 × 0.275 × 0.0198.
(*d*) 2.763 × 59.87 × 0.264 × 0.001702.

Ans.—(*a*) 101,250; (*b*) 10,833; (*c*) 0.000019275; (*d*) 0.074328

Solve the following by the use of logarithms:

(*e*) Find the area of a rectangular lot having a frontage of 126 ft. $6\frac{1}{2}$ in. (equal to 126.54 ft.) and a depth of 286 ft. 3 in. (equal to 286.25 ft.).
Ans. 36,223 sq. ft.

(*f*) What will be the total cost of six lines of copper wire, each 19.36 miles long, if 1,000 feet of wire weighs 49.98 pounds and the cost per pound is $18\frac{3}{4}$ cents? There are 5,280 feet in a mile.
Ans. $5,747.69

DIVISION BY USE OF LOGARITHMS

37. In the division of one indicated power of a number by another indicated power of the same number, the exponent in the quotient is found by subtracting the exponent in the divisor from the exponent in the dividend. Since logarithms are really exponents, it is evident that one number may be divided by another by applying the following rule:

Rule.—*To divide one number by another, first subtract the logarithm of the divisor from the logarithm of the dividend, and thus obtain the logarithm of the quotient. Then, find the number corresponding to that logarithm.*

The logarithm of the dividend should be written first and below it should be written the logarithm of the divisor. In the simplest case of division by means of logarithms, the logarithm of the divisor can be subtracted from the logarithm of the dividend without requiring a change in the form of the logarithm of the dividend, as in the following examples:

EXAMPLE 1.—Divide 6,784.2 by 27.42.

SOLUTION.—The calculations may be indicated in the following manner:

$$\begin{aligned} \log 6{,}784.2 &= 3.83150 \\ \log 27.42 &= \underline{1.43807} \end{aligned}$$

By subtraction, $\log \text{quotient} = 2.39343$

The corresponding number is 247.42. Ans.

EXAMPLE 2.—Divide 0.067842 by 0.002742.

SOLUTION.—The work may be arranged as follows:

$$\begin{aligned} \log 0.067842 &= 8.83150-10 \\ \log 0.002742 &= \underline{7.43807-10} \end{aligned}$$

By subtraction, $\log \text{quotient} = 1.39343$

The corresponding number is 24.742. Ans.

When 10 is subtracted from 10, the result is obviously 0, and therefore no -10 appears in the logarithm of the quotient.

EXAMPLE 3.—Divide 0.8792 by 61.

SOLUTION.—

$$\begin{array}{ll}
\log 0.8792 & = 9.94409-10 \\
\log 61 & = 1.78533 \\
\hline
\text{log quotient} & = 8.15876-10=\log 0.014413
\end{array}$$

Thus, $0.8792 \div 61 = 0.014413$. Ans.

In this case, nothing is subtracted from the 10 in the logarithm of the dividend, and −10 therefore appears in the logarithm of the quotient.

38. In division by means of logarithms, it is sometimes necessary to change the form of the logarithm of the dividend in order to obtain the logarithm of the quotient in suitable form. If the logarithm of the dividend is written first in the usual form and the logarithm of the divisor is written below it, the required change in the form of the logarithm of the dividend becomes obvious.

EXAMPLE 1.—Divide 274.2 by 6,784.2.

SOLUTION.—The work may be arranged as follows:

$$\begin{array}{lll}
\log 274.2 & = 2.43807 & = 12.43807-10 \\
\log 6{,}784.2 & = 3.83150 & = 3.83150 \\
\hline
\text{log quotient} & & = 8.60657-10=\log 0.040417
\end{array}$$

Therefore, $274.2 \div 6{,}784.2 = 0.040417$. Ans.

Before 3.83150 is subtracted from 2.43807, it is necessary to add 10 to 2.43807 and to annex −10, in order to obtain the logarithm of the quotient in positive form. The new form of log 274.2 is 12.43807 − 10.

EXAMPLE 2.—Find the value of $8.792 \div 0.061$.

SOLUTION.—The work is as follows:

$$\begin{array}{lll}
\log 8.792 & = 0.94409 & = 10.94409-10 \\
\log 0.061 & = 8.78533-10 & = 8.78533-10 \\
\hline
\text{log quotient} & & = 2.15876
\end{array}$$

The corresponding number is 144.13. Ans.

Since 0.94409 is smaller than 8.78533, it is necessary to add 10 to 0.94409 and to annex −10. The new form of the logarithm is 10.94409 − 10.

EXAMPLE 3.—Divide 0.00425 by 0.83724.

SOLUTION.—The calculations may be arranged in the following manner:

$$\begin{array}{lcl}
\log 0.00425 = 7.62839-10 & = & 17.62839-20\\
\log 0.83724 = 9.92285-10 & = & 9.92285-10\\
\cline{3-3}
\text{log quotient} & = & 7.70554-10
\end{array}$$

The corresponding number is 0.0050762. Ans.

In this case, the positive part of the logarithm of the dividend, or 7.62839, should be increased by 10 in order that it may exceed the positive part of the logarithm of the divisor or 9.92285. Since the negative part of log 0.00425 must also be increased by 10, the new form of the logarithm is 17.62839 − 20. When the subtraction of log 0.83724 from log 0.00425 is performed, the difference between the negative parts of these two logarithms is found to be 20 − 10 = 10.

EXAMPLE 4.—Divide 879,200 by 0.0000061.

SOLUTION.—

$$\begin{array}{lcl}
\log 879{,}200 \quad = 5.94409 & = & 15.94409-10\\
\log 0.0000061 = 4.78533-10 & = & 4.78533-10\\
\cline{3-3}
\text{log quotient} & = & 11.15876
\end{array}$$

The corresponding number is 144,130,000,000. Ans.

In this case the logarithm of the dividend is greater than the positive part of the logarithm of the divisor. Nevertheless, a −10 is introduced in the logarithm of the dividend in order to balance the −10 in the logarithm of the divisor.

EXAMPLES FOR PRACTICE

Find the quotients of the following by the use of logarithms:

(*a*) 564.35 ÷ 34.96.
(*b*) 9.643 ÷ 200.04.
(*c*) 0.16071 ÷ 76.8.
(*d*) 0.00624 ÷ 3.096.
(*e*) 0.000119 ÷ 0.0719.
(*f*) 1.19 ÷ 719.
(*g*) 1 ÷ 1,728.
(*h*) 1 ÷ 0.7854.

Ans.—(*a*) 16.143; (*b*) 0.048204; (*c*) 0.0020926; (*d*) 0.0020155; (*e*) 0.0016551; (*f*) 0.0016551; (*g*) 0.00057871; (*h*) 1.2732

Solve the following by the use of logarithms:

(*i*) An automobile factory built 12,996 machines at a cost of $8,972,500. What is the average cost of each automobile? Ans. $690.42

(*j*) What is the diameter, in feet, of a circular track 1 mile, or 5,280 feet, in circumference? The diameter is equal to the circumference divided by 3.1416. Ans. 1,680.7 ft.

(*k*) A cubic foot of water at its greatest density weighs 62.425 pounds. What does 1 cubic inch weigh? 1 cubic foot = 1,728 cubic inches. Ans. 0.036126 lb.

FINDING POWERS AND ROOTS BY LOGARITHMS

39. Finding Powers of Numbers.—The operation of raising a number to a power can be conveniently performed by means of logarithms. As previously explained, a number may be raised to an integral power by using it as a factor as many times as is indicated by the exponent. Thus, $2^3=2\times2\times2=8$. If logarithms are used for performing this operation, $\log 2^3=\log 2+\log 2+\log 2=3\log 2=3\times0.30103=0.90309$, which is the logarithm of 8. In a similar manner, $\log 5^4=\log 5+\log 5+\log 5+\log 5=4\log 5=4\times0.69897=2.79588$, which is the logarithm of 625. Also, $\log 3^5=5\log 3=5\times0.47712=2.38560$, which corresponds to the number 243. These are simple illustrations. When logarithms are used, a number may be raised to any integral or fractional power by applying the following rule:

Rule.—*To raise a number to a power, first multiply the logarithm of the number by the exponent that indicates the power to which the number is to be raised, and thus obtain the logarithm of the required result. Then, find the number that corresponds to this logarithm.*

When the given number is greater than 1, the solution is simple, as in the following examples.

EXAMPLE 1.—What is the square of 7.92?

SOLUTION.—The calculations may be carried out as follows:

$$\log 7.92^2=2\log 7.92$$

$$\begin{array}{r}\log 7.92=0.89873\\ 2\\ \hline\end{array}$$

$$\log 7.92^2=1.79746=\log 62.727$$

Thus, $7.92^2=62.727$. Ans.

EXAMPLE 2.—Find the value of $15.987^{0.2883}$.

SOLUTION.—

$$\begin{aligned}\log 15.987^{0.2883}&=0.2883\log 15.987\\ \log 15.987&=1.20377\\ \log 15.987^{0.2883}&=0.2883\times1.20377\\ &=0.34705=\log 2.2236\end{aligned}$$

Hence, $15.987^{0.2883}=2.2236$. Ans.

Only five figures are used in the mantissa of the logarithm of the required result, because only five places are used in the mantissa of the logarithm of the given number.

EXAMPLE 3.—What is the value of $2.5^{\frac{3}{8}}$?

SOLUTION.—

$$\begin{aligned} \log 2.5^{\frac{3}{8}} &= \tfrac{3}{8} \log 2.5 \\ \log 2.5 &= 0.39794 \\ \log 2.5^{\frac{3}{8}} &= \tfrac{3}{8} \times 0.39794 \\ &= 0.14923 = \log 1.41 \end{aligned}$$

Therefore, $2.5^{\frac{3}{8}} = 1.41$. Ans.

40. If a number that is wholly decimal is to be raised to a positive power, both the positive and negative parts of the logarithm of the given number must be multiplied by the exponent indicating the power. The negative part of the resulting logarithm will, therefore, be either greater or smaller than 10, depending on whether the exponent is greater or less than 1. In order to simplify this logarithm, it is necessary to subtract from, or add to, the negative part a number that will change that part to 10. Of course, the same number should also be subtracted from, or added to, the positive part of the logarithm.

EXAMPLE 1.—Raise 0.0751 to the fourth power.

SOLUTION.—$\log 0.0751^4 = 4 \log 0.0751$

$$\begin{array}{rr} \log 0.0751 = & 8.87564 - 10 \\ & 4 \\ \hline \log 0.0751^4 = & 35.50256 - 40 \\ & 30 \quad\quad -30 \\ \hline & 5.50256 - 10 = \log 0.00003181 \end{array}$$

Thus, $0.0751^4 = 0.00003181$. Ans.

In order to change to 10 the negative part of the logarithm $35.50256 - 40$, it is necessary to subtract 30 from each part of the logarithm.

EXAMPLE 2.—Find the value of 0.4486^2.

$$\begin{array}{rr} \text{SOLUTION.—}\log 0.4486 = & 9.65186 - 10 \\ & 2 \\ \hline \log 0.4486^2 = & 19.30372 - 20 \\ & 10 \quad\quad -10 \\ \hline & 9.30372 - 10 = \log 0.20124 \end{array}$$

That is, $0.4486^2 = 0.20124$. Ans.

EXAMPLE 3.—Find the value of $0.83267^{0.8}$.

SOLUTION.—
$$
\begin{array}{ll}
\log 0.83267^{0.8} & =0.8 \log 0.83267\\
\log 0.83267 & =9.92048-10\\
 & 0.8\\
\cline{2-2}
\log 0.83267^{0.8} & =7.93638-8\\
 & 2 \qquad\quad -2\\
\cline{2-2}
 & 9.93638-10=\log 0.86374
\end{array}
$$

Therefore, $0.83267^{0.8}=0.86374$. Ans.

The negative part of the logarithm $7.93638-8$ obtained in this case is less than 10. Therefore, in order to change that part to 10, it is necessary to add $10-8$, or 2, to each part of the logarithm.

EXAMPLE 4.—What is the value of $0.5556^{0.615}$?

SOLUTION.—
$$
\begin{array}{ll}
\log 0.5556^{0.615} & =0.615 \log 0.5556\\
\log 0.5556 & =9.74476-10\\
 & \times 0.615\\
\cline{2-2}
\log 0.5556^{0.615} & =5.99303-6.15\\
 & 3.85 \qquad -3.85\\
\cline{2-2}
 & 9.84303-10 \quad =\log 0.69667
\end{array}
$$

Thus, $0.5556^{0.615}=0.69667$. Ans.

The negative part of the product of log 0.5556 and 0.615 is less than 10 by the amount $10-6.15$, or 3.85. Hence, the logarithm $5.99303-6.15$ is changed to the desired form by increasing each part of it by 3.85.

EXAMPLE 5.—Find the value of $0.002759^{1.25}$.

SOLUTION.—The calculations may be performed in the following manner:

$$
\begin{array}{ll}
\log 0.002759^{1.25} & =1.25 \log 0.002759\\
\log 0.002759 & =7.44075-10\\
 & \times 1.25\\
\cline{2-2}
\log 0.002759^{1.25} & =9.30094-12.5\\
 & 2.5 \qquad\quad -2.5\\
\cline{2-2}
 & 6.80094-10
\end{array}
$$

The corresponding number is 0.00063233. Ans.

In this case, the negative part of the product of log 0.002759 and 1.25 must be reduced by $12.5-10$, or by 2.5, and the positive part is reduced by the same amount.

Example 6.—Find the value of $0.78^{\frac{0.3}{1.3}}$.

Solution.—

$$
\begin{aligned}
\log 0.78^{\frac{0.3}{1.3}} &= \frac{0.3}{1.3} \log 0.78 \\
\log 0.78 &= 9.89209-10 \\
\log 0.78^{\frac{0.3}{1.3}} &= \frac{0.3}{1.3} \times (9.89209-10) \\
&= \frac{0.3}{1.3} \times 9.89209 - \frac{0.3}{1.3} \times 10 \\
&= 2.28279 - 2.30769
\end{aligned}
$$

$$
\begin{array}{l}
7.69231 - 7.69231 \\
\hline
9.97510 - 10
\end{array}
$$

The corresponding number is 0.94428. Ans.

In order to change the logarithm 2.28279−2.30769 to a form in which the negative part is 10, each part is increased by 10−2.30769, or by 7.69231.

41. Extracting Roots of Numbers.—The use of logarithms is especially advantageous in extracting roots of numbers, as it is possible to find rapidly any root of a number by means of logarithms. As previously explained, the process of extracting a root of a number is the reverse of raising a number to a power. In fact, if a root of a given number is raised to the power indicated by an exponent that is equal to the index of the root, the given number is obtained; thus, $\sqrt[3]{8}=2$ and $2^3=8$. Also, since $2^3=8$, $3 \log 2=\log 8$. Hence, if the logarithm of a root of a number is multiplied by the index of the root, the logarithm of the number is obtained. Since $3 \log 2=\log 8$, $\log 2=\frac{\log 8}{3}$. Therefore, it follows that the logarithm of a root of a given number is equal to the logarithm of the number divided by the index of the root. Thus, $\log \sqrt[3]{8}=\frac{\log 8}{3}=\frac{0.90309}{3}=0.30103$, which is the logarithm of 2. Also, $\log \sqrt{9}=\frac{\log 9}{2}=\frac{0.95424}{2}=0.47712$, which is the logarithm of 3. In general, when logarithms are used, any root of a number can be found by applying the following rule:

Rule.—*To extract a root of a number, first divide the logarithm of the number by the index of the root, and thus obtain the logarithm of the required root. Then find the number corresponding to that logarithm.*

When the number is greater than 1, the application of the rule is simple, as in the following examples.

EXAMPLE 1.—Find the square root of 77,851.

SOLUTION.—

$$\log 77{,}851 = 4.89127$$

$$\log \sqrt{77{,}851} = \frac{\log 77{,}851}{2} = \frac{4.89127}{2} = 2.44564$$

The corresponding number is 279.03. Ans

EXAMPLE 2.—Extract the cube root of 698,970.

SOLUTION.—

$$\log 698{,}970 = 5.84446$$

$$\log \sqrt[3]{698{,}970} = \frac{\log 698{,}970}{3} = \frac{5.84446}{3} = 1.94815$$

The corresponding number is 88.746. Ans.

EXAMPLE 3.—Find the value of $\sqrt[2.5]{68.3}$.

SOLUTION.—

$$\log 68.3 = 1.83442$$

$$\log \sqrt[2.5]{68.3} = \frac{\log 68.3}{2.5} = \frac{1.83442}{2.5} = 0.73377$$

The corresponding number is 5.4171. Ans.

42. When it is required to extract a root of a number that is wholly decimal, both the positive part and the negative part of the logarithm of the given number are divided by the index of the root. In solving such a problem, it is customary to express the logarithm of the given number in such a way that the negative part of the quotient will be 10. Therefore, before the division is performed, the negative part of the logarithm of the given number is increased so that it is equal to the product of 10 and the index of the root, and the positive part of the logarithm is increased by the same amount.

EXAMPLE 1.—Extract the square root of 0.0097255.

SOLUTION.—

$$\log \sqrt{0.0097255} = \frac{\log 0.0097255}{2}$$

Since the index of the root is 2, the negative part of the logarithm of the given number should be made equal to 10×2, or 20. The computations for determining the required root are as follows:

$$\begin{array}{rr}
\log 0.0097255 = & 7.98791-10 \\
 & 10 \qquad\quad -10 \\
\hline
 & 2)17.98791-20 \\
\hline
 & 8.99396-10
\end{array}$$

The corresponding number is 0.09862. Ans.

Example 2.—Find the value of $\sqrt[3]{0.0003181}$.

Solution.—In this case, the negative part of the logarithm of the given number should be changed to 10×3, or 30. The work may be arranged in the following manner:

$$\begin{array}{lr}
\log \sqrt[3]{0.0003181} = & \dfrac{\log 0.0003181}{3} \\
\log 0.0003181 \quad = & 6.50256-10 \\
 & 20 \qquad\quad -20 \\
\hline
 & 3)26.50256-30 \\
\hline
 & 8.83419-10 = \log 0.068263
\end{array}$$

That is, $\sqrt[3]{0.0003181} = 0.068263$. Ans.

Example 3.—Find the value of $\sqrt[1.405]{0.888}$.

Solution.—

$$\begin{array}{lr}
\log \sqrt[1.405]{0.888} = & \dfrac{\log 0.888}{1.405} \\
\log 0.888 \quad = & 9.94841-10 \\
 & 4.05 \qquad - \ 4.05 \\
\hline
 & 1.405)13.99841-14.05 \\
\hline
 & 9.96328-10
\end{array}$$

The corresponding number is 0.91892. Ans.

The negative part of log 0.888 should be made equal to 10×1.405, or 14.05. Therefore, 4.05 is added to both parts of the logarithm 9.94841 −10.

EXAMPLES FOR PRACTICE

Find the values of the following by use of logarithms:

(*a*) $1{,}728^2$.

(*b*) 0.64^4.

(*c*) $\sqrt[4]{906.8}$.

(*d*) $\sqrt[5]{11}$.

(*e*) $\sqrt{0.937}$.

(*f*) $\sqrt[3]{0.0497}$.

(*g*) $2{,}350^{0.75}$.

(*h*) $0.03958^{0.875}$.

(*i*) $0.000752^{1.2}$.

(*j*) $0.75^{\frac{3}{2}}$

Ans.—(*a*) 2,985,900; (*b*) 0.16777; (*c*) 5.4876; (*d*) 1.6154; (*e*) 0.96798; (*f*) 0.36766; (*g*) 337.52; (*h*) 0.059266; (*i*) 0.00017843; (*j*) 0.64951

Solve the following by use of logarithms:

(*k*) What is the length, in feet, of each side of a square that has an area of 1 acre, or 43,560 square feet?

Ans. 208.71 ft.

COMBINATION OF OPERATIONS

43. Types of Computations.—The computation of a certain quantity may involve both multiplication and division. Furthermore, some of the factors may contain exponents that are either integral or fractional, and other factors may be roots that are indicated by radical signs. The use of logarithms is especially advantageous in evaluating complex expressions. It should be remembered, however, that logarithms cannot be used for performing addition or subtraction.

44. Calculations Involving Multiplication and Division. The method of procedure to be followed in finding the value of an expression in which both multiplication and division are to be performed, may be illustrated by means of a typical example.

EXAMPLE.—Find the value of k in the formula $k=\frac{abl}{rs}$ when $a=497$, $b=0.0181$, $l=762$, $r=3{,}300$, and $s=0.6517$.

SOLUTION.—The expression abl means $a\times b\times l$, and rs means $r\times s$, the multiplication signs being omitted in the formula in accordance with the usual practice. When the given numerical values are substituted for the letters, the formula becomes

$$k=\frac{497\times 0.0181\times 762}{3{,}300\times 0.6517}$$

and $\log k=(\log 497+\log 0.0181+\log 762)-(\log 3{,}300+\log 0.6517)$

The calculations may be indicated as follows:

$$\begin{array}{ll}
\log 497 & = 2.69636\\
\log 0.0181 & = 8.25768-10\\
\log 762 & = 2.88195\\
\hline
\log (497\times 0.0181\times 762) & =13.83599-10\\
 & = 3.83599\\
 & \\
\log 3{,}300 & = 3.51851\\
\log 0.6517 & = 9.81405-10\\
\hline
\log (3{,}300\times 0.6517) & =13.33256-10\\
 & = 3.33256\\
 & \\
\log (497\times 0.0181\times 762) & = 3.83599\\
\log (3{,}300\times 0.6517) & = 3.33256\\
\hline
\log k & = 0.50343\\
 & k= 3.1874. \quad \text{Ans.}
\end{array}$$

45. Calculations Involving Powers and Roots.—In evaluating an expression that includes indicated powers or roots, it is usually convenient to find first the logarithms of the powers or roots. The required multiplications and divisions should preferably be carried out in the order that permits the most convenient arrangement of logarithmic work and avoids repetition in using values. This is illustrated in the following examples.

Example 1.—In the formula $W=\frac{\pi}{6}D^3w$, which gives the weight of a sphere, W represents the weight; D, the diameter of the sphere; w, the weight of a cubic unit of the material; and π (pronounced pī) $=3.1416$. Find, by the use of logarithms, the value of W, when $D=0.298$ in. and $w=0.3031$ lb.

Solution.—If the given values are substituted for the letters in the formula, the expression obtained is

$$W=\frac{3.1416\times 0.298^3\times 0.3031}{6}$$

from which

$$\log W=\log 3.1416+\log 0.298^3+\log 0.3031-\log 6.$$

The logarithmic work can be arranged most conveniently if the logarithm of 0.298^3 is found first, as follows:

$$\begin{array}{ll}
\log 0.298 & =9.47422-10 \\
\hline
\log 0.298^3=3 \log 0.298 & =8.42266-10 \\
\log 3.1416 & =0.49715 \\
\log 0.3031 & =9.48159-10 \\
\hline
\log (0.298^3\times 3.1416\times 0.3031) & =8.40140-10 \\
\log 6 & =0.77815 \\
\hline
\log & W=7.62325-10 \\
 & W=0.0042 \text{ lb.} \quad \text{Ans.}
\end{array}$$

EXAMPLE 2.—The following formula is used in hydraulics:

$$d=2.57 \sqrt[5]{\frac{flQ^2}{h}}$$

Find d, when $f=0.022$; $l=2{,}820$; $h=133$; and $Q=12.5$.

SOLUTION.—In this case,

$$d=2.57 \sqrt[5]{\frac{0.022\times 2{,}820\times 12.5^2}{133}}$$

and $\log d=\log 2.57+\frac{1}{5}(\log 12.5^2+\log 0.022+\log 2{,}820-\log 133)$

The logarithmic work may be arranged as follows:

$$\begin{array}{ll}
\log 12.5 & =1.09691 \\
\log 12.5^2=2 \log 12.5 & =2.19382 \\
\log 0.022 & =8.34242-10 \\
\log 2{,}820 & =3.45025 \\
\hline
\log (12.5^2\times 0.022\times 2{,}820) & =3.98649 \\
\log 133 & =2.12385 \\
\hline
\log \left(\frac{12.5^2\times 0.022\times 2{,}820}{133}\right) & =1.86264 \\
\frac{1}{5}\times 1.86264 & =0.37253 \\
\log 2.57 & =0.40993 \\
\hline
\log d & =0.78246
\end{array}$$

$d=6.0599$, or 6.06, nearly. Ans.

46. Computations Involving Addition and Subtraction. Where operations that can be performed by means of logarithms are combined with addition and subtraction, care must be taken to carry out the operations in the proper sequence. The following examples serve to illustrate the procedure.

Example 1.—When the perpendicular sides a and b of a right triangle are known, the hypotenuse c can be found by the formula $c=\sqrt{a^2+b^2}$. What is the value of c when $a=119.72$ and $b=98.16$?

Solution.—For the given values of a and b,

$$c=\sqrt{119.72^2+98.16^2}$$

The first step is to find the values of 119.72^2 and 98.16^2 by logarithms, as follows:

$$\begin{array}{rl@{\qquad}rl}
\log 119.72 & =2.07816 & \log 98.16 & =1.99193 \\
 & 2 & & 2 \\
\hline
\log 119.72^2 & =4.15632 & \log 98.16^2 & =3.98386 \\
119.72^2 & =14{,}332 & 98.16^2 & =9{,}635.2, \text{ or } 9{,}635
\end{array}$$

The required value of c is then determined by adding these squares and extracting the square root of the sum by means of logarithms. Thus,

$$14{,}332+9{,}635=23{,}967$$
$$\log 23{,}967=4.37962$$
$$\log c=\log\sqrt{23{,}967}=\tfrac{1}{2}\times 4.37962=2.18981$$
$$c=154.81. \quad \text{Ans.}$$

Example 2.—If the hypotenuse c of a right triangle and one of the other sides a are given, the third side b can be computed by the following formula:

$$b=\sqrt{(c-a)(c+a)}$$

Find the value of b in a triangle in which $c=1{,}331$ and $a=920$.

Solution.—In the given triangle,

$$b=\sqrt{(1{,}331-920)\ (1{,}331+920)}$$

Here, it is first necessary to simplify the expression under the radical sign by performing the indicated subtraction and addition. Thus,

$$1{,}331-920=411$$
$$1{,}331+920=2{,}251$$

Then, the product of these two quantities and the square root of that product are found by logarithms in the following manner:

$$\begin{array}{ll}
\log 411 & =2.61384 \\
\log 2{,}251 & =3.35238 \\
\hline
\log (411\times 2{,}251) & =5.96622 \\
\log b=\tfrac{1}{2}\times 5.96622 & =2.98311 \\
 b & =961.86. \quad \text{Ans.}
\end{array}$$

47. Computations in Which a Logarithm Is a Factor.—In some cases, the logarithm of a quantity, rather than the quantity itself, appears as a factor in a formula. Before the value of the entire expression can be determined, it is necessary to find such a logarithm from a table. The proper substitution can then be made in the formula.

EXAMPLE.—A formula that is used in radio for expressing the ratio of the acoustic powers of two electric currents in terms of the ratio of the electric powers of the currents is

$$N = 10 \log \frac{P_1}{P_2}$$

in which P_1 and P_2 are the electric powers of the two currents and N is the ratio of their acoustic powers. What is the value of N if $\frac{P_1}{P_2} = 3.5$?

SOLUTION.—The first step is to find the value of $\log \frac{P_1}{P_2}$ or $\log 3.5$. From a table of logarithms, this is 0.54407. The value of N is then equal to $10 \times 0.54407 = 5.4407$, or say 5.44. Ans.

48. Accuracy of Results Obtained by Use of Five-Place Logarithmic Table.—When five-place logarithms are used in performing calculations, the results obtained are correct to four significant figures, and the fifth significant figure may or may not be exactly correct. There are various reasons why the fifth figure may not be accurate. As previously explained, mantissas are unending decimals and the values printed in a five-place table are approximate. When the mantissas for such a table are computed, all figures beyond the fifth figure are dropped and the fifth figure is increased by 1 if the sixth figure would be 5 or more. Furthermore, when the fifth figure of the mantissa for a given number is determined by interpolation, that figure may be in error by 1, because the proportional part is approximate and the inaccuracy in the proportional part added to the inaccuracy in the printed mantissa may exceed 0.000005. Also, when the logarithms of two or more numbers are added, as when the product of these numbers is required, the sum of the inaccuracies in the individual logarithms may exceed 0.000005, and there will then be an error in the fifth figure of the required logarithm. In some cases, on the other hand, the errors in two logarithms used in a computation may tend to offset each other, and the final result may therefore be correct.

In general, it should not be expected that the fifth figure in a result obtained by the use of a five-place logarithmic table will be exactly correct. Hence, if a problem is solved both by arithmetic and by logarithms, it may be found that the results differ in the fifth significant figure.

The nature of the computations determines what degree of accuracy is required and what kind of logarithmic table is to be used. For most engineering computations, a five-place table provides ample accuracy. There are, however, certain computations in precise surveying work for which a five-place table is not sufficiently accurate and for which it is necessary to use a six- or seven-place table. For astronomical computations, even a seven-place table is not accurate enough, and it is customary to use a ten-place table. The general methods here explained for computations with a five-place table apply also to computations with larger and more accurate tables of logarithms.

EXAMPLES FOR PRACTICE

Solve the following by means of logarithms:

1. Find the value of a in the formula $a=\frac{b\times c\times d}{e\times f}$, if $b=89$, $c=753$, $d=0.0097$, $e=36{,}709$, and $f=0.08497$.

Ans. 0.20840, or 0.2084

2. The formula used in mechanics for finding the velocity of a falling body is $v=\sqrt{2gh}$, in which v is the velocity at any instant, g is the increase in velocity in each second, and h is the height through which the body has fallen. Find v when $g=32.16$ and $h=220$. Ans. 118.96

3. The area of an equilateral triangle may be determined by the formula $A=\frac{l^2}{4}\times\sqrt{3}$, in which A denotes the area of the triangle and l the length of each side. What is the value of A when $l=276.3$?

Ans. 33,057

4. The formula $H=\frac{PLAN}{33{,}000}$ is used to compute the horsepower of a steam or gas engine. In this formula, H is the horsepower; P, the average effective pressure on the piston, in pounds per square inch; L, the distance the piston travels per stroke, in feet; A, the area of cross-section of the

cylinder, in square inches; and N, the number of working strokes per minute. Find H, when $P=62.76$ pounds; $L=31\frac{1}{2}$ inches$=2.625$ feet; $A=18.42$ square inches; and $N=92.4$. Ans. 8.4968, or 8.5

Suggestion: $\log H=\log P+\log L+\log A+\log N-\log 33{,}000$.

5. The following formula, given in the subject of Heat, is used to find the absolute temperature of a gas expanding under certain conditions:

$$T_2=T_1\times\left(\frac{P_2}{P_1}\right)^{\frac{0.405}{1.405}}$$

in which T_1 and T_2 represent temperatures, and P_1 and P_2 corresponding pressures. Find T_2, when $T_1=520$, $P_2=235$, and $P_1=14.7$.
Ans. 1,156.1, or 1,156

Suggestion: $\log T_2=\log T_1+\frac{0.405}{1.405}(\log P_2-\log P_1)$.

COLOGARITHMS

49. Use of Cologarithms.—In evaluating certain types of expressions, the logarithmic work can be arranged more conveniently if subtraction of logarithms is avoided. This can be readily accomplished because dividing by a number is the same as multiplying by its reciprocal, which is 1 divided by the number. Thus, instead of dividing 28.97 by 7.354, the same result may be obtained by multiplying 28.97 by $\frac{1}{7.354}$.

The logarithm of the reciprocal of a number is called the *cologarithm* of the number. The abbreviation *colog* is used for the expression *the cologarithm of*. Thus, the cologarithm of x is indicated simply as colog x, and its value is the same as that of $\log\frac{1}{x}$. Since $\log\left(\frac{1}{x}\right)=\log 1-\log x$ and log 1 equals zero, it follows that $\text{colog } x=0-\log x=10-\log x-10$. Hence, the cologarithm of any number can be found from the logarithm of the number by subtraction.

If the given number is greater than 1, its cologarithm is obtained by subtracting its logarithm from 10 and annexing -10 to the remainder. For example, $\text{colog } 3{,}300=10-\log 3{,}300-10=10-3.51851-10=6.48149-10$. If the given number is wholly decimal, its logarithm will contain a -10 and its cologarithm may be obtained merely by subtracting the positive part of its logarithm from 10, because the -10 in the minuend will cancel the −10 in the subtrahend. Thus, since $\log 0.6517 = 9.81405 - 10$, $\operatorname{colog} 0.6517 = 10 - 9.81405 = 0.18595$. The cologarithm of a number can usually be determined from the logarithm of the number by a mental calculation. To perform the calculation, the last significant figure of the logarithm is subtracted from 10, and each other figure of the logarithm is subtracted from 9.

50. Computations With Cologarithms.—The manner in which the logarithmic work can be arranged when cologarithms are used may be indicated by considering again the values in the example of Art. 44. When division is replaced by multiplication by a reciprocal and the numerical values are used, the formula may be written as follows:

$$k = 497 \times 0.0181 \times 762 \times \frac{1}{3{,}300} \times \frac{1}{0.6517}$$

Therefore,

$$\log k = \log 497 + \log 0.0181 + \log 762 + \operatorname{colog} 3{,}300 + \operatorname{colog} 0.6517$$

The logarithms may then be tabulated in the following manner:

$$\begin{aligned}
\log 497 &= 2.69636 \\
\log 0.0181 &= 8.25768 - 10 \\
\log 762 &= 2.88195 \\
\operatorname{colog} 3{,}300 &= 6.48149 - 10 \\
\operatorname{colog} 0.6517 &= 0.18595 \\
\hline
\log k &= 20.50343 - 20 \\
&= 0.50343 \quad = \log 3.1874
\end{aligned}$$

HYPERBOLIC LOGARITHMS

51. In the system of logarithms used in the preceding articles, the base is the number 10, and all logarithms are exponents of the number 10. The distinguishing names for these logarithms are *common logarithms* and *Briggs logarithms*, but they are usually called simply *logarithms*. They are generally used for the computations in practice. However, any positive number except 1 can be used as a base

In certain special computations that occasionally come up in scientific work, use is made of a system of logarithms in which the number 2.718281828+ is the base. This number is generally denoted by the letter e. Logarithms to the base e are known as *hyperbolic*, *Naperian*, or *natural logarithms*. In practical computations, it is customary to designate the hyperbolic logarithm of a number by writing the subscript e after the abbreviation log. Thus, $\log_e 4$ means the hyperbolic logarithm of 4. When no base is designated, common logarithms to the base 10 are usually meant.

52. For computations that require the use of hyperbolic, or natural, logarithms, special tables of such logarithms may be employed. When such special tables are not available, the hyperbolic logarithm of any number may be determined by first finding the common logarithm (characteristic and mantissa) of the given number and then multiplying that complete logarithm by the constant 2.3026. If a less accurate value is satisfactory, the complete common logarithm may be multiplied by 2.303 or even by 2.3. For example, the hyperbolic logarithm of 20, or $\log_e 20$, may be determined as follows:

$$\log 20 = 1.30103$$

$$\log_e 20 = 2.3026 \times 1.30103 = 2.99575$$

Conversely, if the hyperbolic logarithm of a number is known and it is required to find the number without the use of a special table, the first step is to divide the hyperbolic logarithm by 2.3026 or to multiply that logarithm by 0.4343, which is the reciprocal of 2.3026. The result is the common logarithm of the required number, and that number can be readily found. Thus, the number N whose hyperbolic logarithm is 3.47182 may be found in the following manner:

$$\log_e N = 3.47182$$

$$\log N = 0.4343 \times 3.47182 = 1.50781 = \log 32.196$$

W7-CMM-092